

# FRAMED MOTIVES OF RELATIVE MOTIVIC SPHERES

GRIGORY GARKUSHA, ALEXANDER NESHITOV, AND IVAN PANIN

**ABSTRACT.** The category of framed correspondences  $\mathrm{Fr}_*(k)$ , framed presheaves and framed sheaves were invented by Voevodsky in his unpublished notes [Voe]. Based on the theory, framed motives are introduced and studied in [GP1]. The aim of this paper is to prove the following results stated in [GP1, 9.3]: for any  $k$ -smooth scheme  $X$  and any  $n \geq 1$  the map of simplicial pointed sheaves  $(-, \mathbb{A}^1] \mathbb{G}_m)_+^{\wedge n} \rightarrow T^n$  induces a Nisnevich local level weak equivalence of  $S^1$ -spectra

$$M_{fr}(X \times (\mathbb{A}^1] \mathbb{G}_m)^{\wedge n}) \rightarrow M_{fr}(X \times T^n)$$

and the sequence of  $S^1$ -spectra

$$M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow M_{fr}(X \times T^n \times \mathbb{A}^1) \rightarrow M_{fr}(X \times T^{n+1})$$

is locally a homotopy cofiber sequence in the Nisnevich topology.

## CONTENTS

1. Introduction	1
2. Framed presheaves $\mathrm{Fr}(-, Y/(Y-S))$ and $\mathbb{Z}\mathrm{F}(-, Y/(Y-S))$	3
3. Presheaves $\mathrm{Fr}^{qf}(-, Y/(Y-S) \wedge T)$ and $\mathbb{Z}\mathrm{F}^{qf}(-, Y/(Y-S) \wedge T)$	5
4. Useful lemmas	7
5. The linear framed presheaf $\mathbb{Z}\mathrm{F}^{qf}(X_+ \wedge T^{n+1})$	8
6. A filtration on $\mathbb{Z}\mathrm{F}_n(-, X_+ \wedge T^{n+1})$	11
7. Moving lemma	13
8. Proof of Theorem 1.1	16
Appendix A.	19
Appendix B.	20
References	23

## 1. INTRODUCTION

Based on Voevodsky's theory of framed correspondences [Voe], the machinery of framed motives has been developed in [GP1]. One of the main purposes of the machinery is to compute an explicit fibrant resolution of the suspension  $\mathbb{P}^1$ -spectrum  $\Sigma_{\mathbb{P}^1}^\infty X_+$  of a  $k$ -smooth algebraic variety  $X \in \mathrm{Sm}/k$  [GP1, 9.5].

As it is shown in [GP1], the category of framed correspondences of level zero  $\mathrm{Fr}_0(k)$  has an action by pointed sets  $X \otimes K := \bigsqcup_{K \setminus *} X$  with  $X \in \mathrm{Sm}/k$  and  $K$  a pointed set. The cone of  $X$  is

---

2010 *Mathematics Subject Classification.* 14F42, 55P42.

*Key words and phrases.* Motivic homotopy theory, framed motives, motivic spheres.

This paper was partly written during the visit of the second author to Swansea University. He would like to thank the University for the kind hospitality. The third author thanks for the support the Russian Science Foundation (grant no. 14-21-00035).

the simplicial object  $X \otimes I$  in  $\mathrm{Fr}_0(k)$ , where  $(I, 1)$  is the pointed simplicial set  $\Delta[1]$  with basepoint 1. There is a natural morphism  $i_0 : X \rightarrow X \otimes I$  in  $\Delta^{\mathrm{op}} \mathrm{Fr}_0(k)$ . Given an inclusion of smooth schemes  $X \hookrightarrow Y$ , denote by  $Y \downarrow X$  a simplicial object in  $\mathrm{Fr}_0(k)$  which is obtained from the pushout in  $\Delta^{\mathrm{op}} \mathrm{Fr}_0(k)$  of the diagram

$$Y \hookleftarrow X \xrightarrow{i_0} X \otimes I.$$

We can think of  $Y \downarrow X$  as a cone of the inclusion  $X \hookrightarrow Y$ . It is the underlying simplicial scheme for the ordinary pointed motivic space  $(-, Y \downarrow X)_+$ . Proceeding inductively in this way, we define a  $n$ -multisimplicial scheme  $(Y \downarrow X)^{\wedge n}$ , which is also the successive cone of the standard  $n$ -dimensional cube  $(X \hookrightarrow Y)^{\times n}$  in  $\mathrm{Fr}_0(k)$ . The diagonal of  $(Y \downarrow X)^{\wedge n}$  is the underlying simplicial scheme for the motivic space  $(-, Y \downarrow X)_+^{\wedge n} := (-, Y \downarrow X)_+ \wedge \cdots \wedge (-, Y \downarrow X)_+$ .

The computation of an explicit fibrant resolution of  $\Sigma_{\mathbb{P}^1}^\infty X_+$  requires the following theorem (we refer the reader to [GP1, Def. 9.1] for the relevant definitions of framed motives of simplicial Nisnevich sheaves).

**Theorem 1.1.** *Let  $k$  be an infinite perfect field. For any  $k$ -smooth scheme  $X \in \mathrm{Sm}/k$  and any  $n \geq 1$ , the map of simplicial pointed sheaves  $(-, \mathbb{A}^1 \downarrow \mathbb{G}_m)_+^{\wedge n} \rightarrow T^n$  induces a level Nisnevich local weak equivalence of  $S^1$ -spectra*

$$M_{fr}(X \times (\mathbb{A}^1 \downarrow \mathbb{G}_m)^{\wedge n}) \rightarrow M_{fr}(X \times T^n).$$

Moreover, the sequence of  $S^1$ -spectra

$$M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow M_{fr}(X \times T^n \times \mathbb{A}^1) \rightarrow M_{fr}(X \times T^{n+1})$$

is locally a homotopy cofiber sequence in the Nisnevich topology.

The main goal of the paper is to prove this theorem. In particular, it computes locally the stable homotopy type of the framed motive  $M_{fr}(X \times T^n)$  of the relative motivic sphere  $X_+ \wedge T^n = X \times \mathbb{A}^n / (X \times \mathbb{A}^n - X \times 0)$  as the framed motive  $M_{fr}(X \times (\mathbb{A}^1 \downarrow \mathbb{G}_m)^{\wedge n})$  of the multisimplicial scheme  $X \times (\mathbb{A}^1 \downarrow \mathbb{G}_m)^{\wedge n}$ . Another consequence of the theorem says that for every  $n \geq 0$  the natural morphism

$$M_{fr}(X \times T^n \times (\mathbb{A}^1 \downarrow \mathbb{G}_m)) \rightarrow M_{fr}(X \times T^{n+1}) \quad (1)$$

is locally a level weak equivalence of  $S^1$ -spectra in the Nisnevich topology (see Corollary 8.1). We should stress that the proof of [GP1, 9.5] depends on the local equivalence (1).

It is worth to mention that the framed motive functor defined in [GP1, Def. 9.1] converts motivic equivalences between pointed projectively cofibrant motivic spaces to Nisnevich local stable equivalences of  $S^1$ -spectra. However, it is an interesting open question whether the functor converts motivic equivalences between *any* pointed motivic spaces to Nisnevich local stable equivalences. Theorem 1.1 gives an affirmative answer to this question for the motivic equivalence  $X_+ \wedge (-, \mathbb{A}^1 \downarrow \mathbb{G}_m)_+^{\wedge n} \rightarrow X_+ \wedge T^n$  (its proof is in Section 8).

The following theorem (see Theorem 1.2) is *crucial* in our analysis and allows to reduce many computations for framed motives of algebraic varieties to analogous computations for complexes of linear framed presheaves, which are normally much simpler. This is the case, in particular, with Theorem 1.1. It reduces to Theorem 1.3. It is as well worth to mention another similar application of this kind. In [AGP] the Cancellation Theorem for framed motives of algebraic varieties is proved by reducing it to complexes of linear framed presheaves.

**Theorem 1.2.** *For any integer  $m \geq 0$ , the natural morphism of framed  $S^1$ -spectra*

$$\lambda_{X_+ \wedge T^m} : \mathbb{Z} \mathrm{Fr}_*^{S^1}(X_+ \wedge T^m) \rightarrow EM(\mathbb{Z} \mathrm{F}_*(-, X_+ \wedge T^m))$$

is a schemewise stable equivalence. Moreover, the natural morphism of framed  $S^1$ -spectra

$$l_{X \times T^m} : \mathbb{Z}M_{fr}(X \times T^m) \rightarrow LM_{fr}(X \times T^m)$$

is a schemewise stable equivalence. In particular, for any  $U \in Sm/k$  one has

$$\pi_*(\mathbb{Z}M_{fr}(X \times T^m)(U)) = H_*(\mathbb{Z}F(\Delta^\bullet \times U, X_+ \wedge T^m)) = H_*(C_*\mathbb{Z}F(U, X_+ \wedge T^m)).$$

Notation used in Theorem 1.2 is explained in Section 8. The theorem itself is proved in Appendix B. Thus in order to prove Theorem 1.1, it is *sufficient* to prove the following result.

**Theorem 1.3.** *Let  $k$  be an infinite perfect field. For any  $k$ -smooth scheme  $X$  and any  $n \geq 0$  the natural map of complexes of linear framed presheaves*

$$C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1)/C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow C_*\mathbb{Z}F(X_+ \wedge T^{n+1})$$

*is a Nisnevich local equivalence.*

Our proof of Theorem 1.3 splits in two steps, each of which is of independent interest. Firstly we introduce a linear framed sub-presheaf  $\mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$  of the linear framed presheaf  $\mathbb{Z}F(X_+ \wedge T^{n+1})$  and prove the following

**Theorem 1.4.** *For any  $k$ -smooth scheme  $X$  and any  $n \geq 0$ , the natural morphism*

$$\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1)/\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow \mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$$

*of linear framed presheaves is an isomorphism locally in the Nisnevich topology.*

Applying the singular complex construction and the Resolution Theorem of [GP1], one can show that the natural morphism of complexes of linear framed presheaves

$$C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1)/C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow C_*\mathbb{Z}F^{qf}(X_+ \wedge T^{n+1}) \quad (2)$$

is locally a quasi-isomorphism in the Nisnevich topology (see Proposition 5.5 for details).

Secondly using a moving lemma discussed in Section 7, we then prove the following

**Theorem 1.5.** *The inclusion of complexes of linear framed presheaves*

$$C_*\mathbb{Z}F^{qf}(X_+ \wedge T^{n+1}) \hookrightarrow C_*\mathbb{Z}F(X_+ \wedge T^{n+1})$$

*is locally a quasi-isomorphism in the Zarisky topology.*

Clearly, Theorem 1.5 together with the Nisnevich local quasi-isomorphism (2) imply Theorem 1.3.

Throughout the paper we denote by  $Sm/k$  the category of smooth separated schemes of finite type over the base field  $k$ . The subcategory of affine smooth  $k$ -varieties is denoted by  $AffSm/k$ . Given a scheme  $W$  and a family of regular function  $\varphi_1, \dots, \varphi_m$  on  $W$ , we write  $Z(\varphi_1, \dots, \varphi_m)$  to denote a closed subset in  $W$  which is the common vanishing locus of the family  $\varphi_1, \dots, \varphi_m$ . Whenever we speak about  $\Gamma$ -spaces we follow the terminology of Bousfield–Friedlander [BF].

## 2. FRAMED PRESHEAVES $\text{Fr}(-, Y/(Y - S))$ AND $\mathbb{Z}F(-, Y/(Y - S))$

**Definition 2.1.** (I) Let  $Y$  be a  $k$ -smooth scheme and  $S \subset Y$  be a closed subset and let  $U \in Sm/k$ . An explicit framed correspondence of level  $m \geq 0$  from  $U$  to  $Y/(Y - S)$  consists of the following tuples:

$$(Z, W, \varphi_1, \dots, \varphi_m; g : W \rightarrow Y),$$

where  $Z$  is a closed subset of  $U \times \mathbb{A}^m$ , finite over  $U$ ,  $W$  is an étale neighborhood of  $Z$  in  $U \times \mathbb{A}^m$ ,  $\varphi_1, \dots, \varphi_m$  are regular functions on  $W$ ,  $g$  is a regular map such that  $Z = Z(\varphi_1, \dots, \varphi_m) \cap$

$g^{-1}(S)$ . The set  $Z$  is called the *support* of the explicit framed correspondence. We shall also write quadruples  $\Phi = (Z, W, \varphi; g)$  to denote explicit framed correspondences.

(II) Two explicit framed correspondences  $(Z, W, \varphi; g)$  and  $(Z', W', \varphi'; g')$  of level  $m$  are said to be *equivalent* if  $Z = Z'$  and there exists an étale neighborhood  $W''$  of  $Z$  in  $W \times_{\mathbb{A}_U^m} W'$  such that  $\varphi \circ pr$  agrees with  $\varphi' \circ pr'$  and the morphism  $g \circ pr$  agrees with  $g' \circ pr'$  on  $W''$ .

(III) A *framed correspondence of level  $m$  from  $U$  to  $Y/(Y - S)$*  is the equivalence class of an explicit framed correspondence of level  $m$  from  $U$  to  $Y/(Y - S)$ . We write  $\text{Fr}_m(U, Y/(Y - S))$  to denote the set of framed correspondences of level  $m$  from  $U$  to  $Y/(Y - S)$ . We regard it as a pointed set whose distinguished point is the class  $0_{Y/(Y - S), m}$  of the explicit correspondence  $(Z, W, \varphi; g)$  with  $W = \emptyset$ .

(IV) If  $S = Y$  then the set  $\text{Fr}_m(U, Y/(Y - S))$  is denoted by  $\text{Fr}_m(U, Y)$  and is called the *set of framed correspondences of level  $m$  from  $U$  to  $Y$* .

(V) Following Voevodsky [Voe], the *category of framed correspondences*  $\text{Fr}_*(k)$  has objects those of  $Sm/k$  and its morphisms are the sets  $\text{Fr}_*(U, Y) := \bigsqcup_{m \geq 0} \text{Fr}_m(U, Y)$ ,  $U, Y \in Sm/k$ .

(VI) A *framed presheaf* is just a contravariant functor from the category  $\text{Fr}_*(k)$  to sets.

Let  $X, Y$  and  $S$  be  $k$ -smooth schemes and let

$$\Psi = (Z', \mathbb{A}^k \times V \xleftarrow{(\alpha, \pi')} W', \psi_1, \psi_2, \dots, \psi_k; g : W' \rightarrow U) \in \text{Fr}_k(V, U)$$

be an explicit correspondence of level  $k$  from  $V$  to  $U$  and let

$$\Phi = (Z, \mathbb{A}^m \times U \xleftarrow{(\beta, \pi)} W, \varphi_1, \varphi_2, \dots, \varphi_m; g' : W \rightarrow Y) \in \text{Fr}_m(U, Y/(Y - S))$$

be an explicit correspondence of level  $m$  from  $U$  to  $Y/(Y - S)$ . We define  $\Psi^*(\Phi)$  as an explicit correspondence of level  $k + m$  from  $V$  to  $Y/(Y - S)$  as

$$(Z \times_U Z', \mathbb{A}^{k+m} \times V \xleftarrow{(\alpha, \beta, \pi')} W' \times_U W, \psi_1, \psi_2, \dots, \psi_k, \varphi_1, \varphi_2, \dots, \varphi_m, g' \circ pr_W) \in \text{Fr}_{k+m}(V, Y/(Y - S)).$$

Clearly, the pull-back operation  $(\Psi, \Phi) \mapsto \Psi^*(\Phi)$  of explicit correspondences respects the equivalence relation on them. We get a pairing

$$\text{Fr}_k(V, U) \times \text{Fr}_m(U, Y/(Y - S)) \rightarrow \text{Fr}_{k+m}(V, Y/(Y - S)) \quad (3)$$

making  $\text{Fr}_*(U, Y/(Y - S)) := \bigsqcup_{m \geq 0} \text{Fr}_m(-, Y/(Y - S))$  a  $\text{Fr}_*(k)$ -presheaf.

Let  $X, Y, S$  and  $T$  be smooth schemes. There is an *external product*

$$\text{Fr}_m(U, Y/(Y - S)) \times \text{Fr}_n(pt, pt) \xrightarrow{- \boxtimes -} \text{Fr}_{m+n}(U, Y/(Y - S)) \quad (4)$$

given by

$$((Z, W, \varphi_1, \varphi_2, \dots, \varphi_m; g), (Z', W', \psi_1, \dots, \psi_n)) \mapsto (Z \times Z', W \times W', \varphi_1, \varphi_2, \dots, \varphi_m, \psi_1, \dots, \psi_n; g).$$

Set  $\sigma := (\{0\}, \mathbb{A}^1, id : \mathbb{A}^1 \rightarrow \mathbb{A}^1, const : \mathbb{A}^1 \rightarrow pt) \in \text{Fr}_1(pt, pt)$ . Denote by

$$\Sigma : \text{Fr}_m(U, Y/(Y - S)) \rightarrow \text{Fr}_{m+1}(U, Y/(Y - S))$$

the map  $\Phi \mapsto \Phi \boxtimes \sigma$ . Following Voevodsky [Voe] we give the following

**Definition 2.2.** We shall refer to the set

$$\text{Fr}(U, Y/(Y - S)) := \text{colim}(\text{Fr}_0(U, Y/(Y - S)) \xrightarrow{\Sigma} \text{Fr}_1(U, Y/(Y - S)) \xrightarrow{\Sigma} \text{Fr}_2(U, Y/(Y - S)) \dots)$$

as the *set stable framed correspondences from  $U$  to  $Y/(Y - S)$* . Clearly,  $\text{Fr}(-, Y/(Y - S))$  is a framed presheaf of pointed sets with the empty framed correspondence being the distinguished point.

Clearly,  $\text{Fr}(-, Y/(Y-S))$  is even a framed functor in the sense of Voevodsky [Voe] meaning that  $\text{Fr}(\emptyset) = *$  and  $\text{Fr}(U_1 \sqcup U_2, Y/(Y-S)) = \text{Fr}(U_1, Y/(Y-S)) \times \text{Fr}(U_2, Y/(Y-S))$ .

**Definition 2.3** (cf. [GP1, GP3]). Let  $Y \in \text{Sm}/k$  and  $S \subset Y$  be as in Definition 2.1. Let  $U$  be a  $k$ -smooth scheme. Denote by

- ◊  $\mathbb{Z}\text{Fr}_m(U, Y/(Y-S)) := \widetilde{\mathbb{Z}}[\text{Fr}_m(U, Y/(Y-S))] = \mathbb{Z}[\text{Fr}_m(U, Y/(Y-S))]/\mathbb{Z} \cdot 0_{Y/(Y-S), m}$ , i.e the free abelian group generated by the set  $\text{Fr}_m(U, Y/(Y-S))$  modulo  $\mathbb{Z} \cdot 0_{Y/(Y-S), m}$ ;
- ◊  $\mathbb{Z}\text{F}_m(U, Y/(Y-S)) := \mathbb{Z}\text{Fr}_m(U, Y/(Y-S))/A$ , where  $A$  is the subgroup generated by the elements

$$(Z \sqcup Z', W, (\varphi_1, \varphi_2, \dots, \varphi_m); g) - \\ - (Z, W \setminus Z', (\varphi_1, \varphi_2, \dots, \varphi_m)|_{W \setminus Z'}; g|_{W \setminus Z'}) - (Z', W \setminus Z, (\varphi_1, \varphi_2, \dots, \varphi_m)|_{W \setminus Z}; g|_{W \setminus Z}).$$

The elements of  $\mathbb{Z}\text{F}_m(U, Y/(Y-S))$  are called *linear framed correspondences from  $U$  to  $Y/(Y-S)$  of level  $m$* .

**Definition 2.4.** Define  $\text{F}_m(U, Y/(Y-S)) \subset \text{Fr}_m(U, Y/(Y-S))$  as a subset consisting of  $(Z, W, \varphi; g) \in \text{Fr}_m(U, Y/(Y-S))$  such that  $Z$  is connected.

Clearly, the set  $\text{F}_m(U, Y/(Y-S)) - 0_m$  is a free basis of the abelian group  $\mathbb{Z}\text{F}_m(U, Y/(Y-S))$ . However, the assignment  $U \mapsto \text{F}_m(U, Y/(Y-S))$  is not a presheaf even on the category  $\text{Sm}/k$ .

The category of *linear framed correspondences*  $\mathbb{Z}\text{F}_*(k)$  is defined in [GP1]. We shall also refer to contravariant functors from the category  $\mathbb{Z}\text{F}_*(k)$  to Abelian groups as *linear framed presheaves*.

Set  $\mathbb{Z}\text{F}_*(U, Y/(Y-S)) = \bigoplus_{m \geq 0} \mathbb{Z}\text{F}_m(U, Y/(Y-S))$ . The pairing (3) induces in a natural way a bilinear pairing

$$\mathbb{Z}\text{F}_k(V, U) \times \mathbb{Z}\text{F}_m(U, Y/(Y-S)) \rightarrow \mathbb{Z}\text{F}_{k+m}(U, Y/(Y-S)). \quad (5)$$

The latter pairing makes  $\mathbb{Z}\text{F}_*(-, Y/(Y-S))$  a linear framed presheaf.

The external product (4) induces in a natural way an external product of the form

$$\mathbb{Z}\text{F}_m(U, Y/(Y-S)) \times \mathbb{Z}\text{F}_n(\text{pt}, \text{pt}) \xrightarrow{-\boxtimes-} \mathbb{Z}\text{F}_{m+n}(U, Y/(Y-S)) \quad (6)$$

Let  $Y \in \text{Sm}/k$  and  $S \subset Y$  be as in Definition 2.1. *One of the main linear framed presheaves of this paper* we are interested in is defined as

$$\mathbb{Z}\text{F}(-, Y/(Y-S)) = \text{colim}(\mathbb{Z}\text{F}_0(-, Y/(Y-S)) \xrightarrow{\Sigma} \mathbb{Z}\text{F}_1(-, Y/(Y-S)) \xrightarrow{\Sigma} \mathbb{Z}\text{F}_2(-, Y/(Y-S)) \xrightarrow{\Sigma} \dots).$$

### 3. PRESHEAVES $\text{Fr}^{af}(-, Y/(Y-S) \wedge T)$ AND $\mathbb{Z}\text{F}^{af}(-, Y/(Y-S) \wedge T)$

Let  $Y$  be a  $k$ -smooth variety,  $S \subset Y$  be a closed subset. Then the Nisnevich sheaf  $Y/(Y-S) \wedge T$  equals to  $Y \times \mathbb{A}^1/(Y \times \mathbb{A}^1 - S \times \{0\})$ . Thus we have framed presheaves

$$\text{Fr}_*(-, Y/(Y-S) \wedge T) := \text{Fr}_*(-, Y \times \mathbb{A}^1/(Y \times \mathbb{A}^1 - S \times \{0\})) \quad \text{and} \\ \text{Fr}(-, Y/(Y-S) \wedge T) := \text{Fr}(-, Y \times \mathbb{A}^1/(Y \times \mathbb{A}^1 - S \times \{0\}))$$

and linear framed presheaves

$$\mathbb{Z}\text{F}_*(-, Y/(Y-S) \wedge T) := \mathbb{Z}\text{F}_*(-, Y \times \mathbb{A}^1/(Y \times \mathbb{A}^1 - S \times \{0\})) \quad \text{and} \\ \mathbb{Z}\text{F}(-, Y/(Y-S) \wedge T) := \mathbb{Z}\text{F}(-, Y \times \mathbb{A}^1/(Y \times \mathbb{A}^1 - S \times \{0\})).$$

Specifying definitions of the previous section, a section of  $\text{Fr}_m(-, Y/(Y-S) \wedge T)$  on  $U \in \text{Sm}/k$  is given by a tuple

$$(Z, W, \varphi_1, \dots, \varphi_m; g : W \rightarrow Y; f : W \rightarrow \mathbb{A}^1),$$

where  $Z$  is a closed subset of  $U \times \mathbb{A}^m$  finite over  $U$ ,  $W$  is an étale neighborhood of  $Z$  in  $U \times \mathbb{A}^m$ ,  $\varphi_1, \dots, \varphi_m, f$  are regular functions on  $W$ ,  $g$  is a regular map such that

$$Z = Z(\varphi_1, \dots, \varphi_m, f) \cap g^{-1}(S).$$

The suspension map

$$\Sigma : \text{Fr}_m(-, Y/(Y-S) \wedge T) \rightarrow \text{Fr}_{m+1}(-, Y/(Y-S) \wedge T)$$

sends  $(Z, W, \varphi_1, \dots, \varphi_m; g; f)$  to  $(Z \times \{0\}, W \times \mathbb{A}^1, \varphi_1 \circ pr_W, \dots, \varphi_m \circ pr_W, pr_{\mathbb{A}^1}; g \circ pr_W; f \circ pr_W)$ . For brevity, we write  $(Z \times \{0\}, W \times \mathbb{A}^1, \varphi, t; g; f)$  for the latter framed correspondence.

Let  $\text{Fr}_m^{qf}(U, Y/(Y-S) \wedge T) \subset \text{Fr}_m(U, Y/(Y-S) \wedge T)$  be a subset consisting of those elements  $c$  for which there is an explicit framed correspondence  $(Z, W, \varphi_1, \dots, \varphi_m; g; f)$  representing  $c$  such that the closed subset  $Z(\varphi_1, \dots, \varphi_m) \cap g^{-1}(S) \subset W$  is *quasi-finite* over  $U$ .

Set  $\text{Fr}_*^{qf}(U, Y/(Y-S) \wedge T) := \bigsqcup_{m \geq 0} \text{Fr}_m^{qf}(U, Y/(Y-S) \wedge T)$ . Clearly,  $\text{Fr}_*^{qf}(-, Y/(Y-S) \wedge T)$  is a framed subpresheaf of the framed presheaf  $\text{Fr}_*(-, Y/(Y-S) \wedge T)$ . Also, it is clear that the suspension  $\Sigma$  takes  $\text{Fr}_m^{qf}(U, Y/(Y-S) \wedge T)$  to  $\text{Fr}_{m+1}^{qf}(U, Y/(Y-S) \wedge T)$ . Set

$$\text{Fr}^{qf}(-, Y/(Y-S) \wedge T) := \text{colim}(\text{Fr}_0^{qf}(-, Y/(Y-S) \wedge T) \xrightarrow{\Sigma} \text{Fr}_1^{qf}(-, Y/(Y-S) \wedge T) \xrightarrow{\Sigma} \dots)$$

By the very construction,  $\text{Fr}^{qf}(-, Y/(Y-S) \wedge T)$  is a pointed framed subpresheaf of the pointed framed presheaf  $\text{Fr}(-, Y/(Y-S) \wedge T)$ .

Let  $(Z, W, \varphi; g; f) \in \text{Fr}_m(U, Y/(Y-S))$  be such that  $Z = Z_1 \sqcup Z_2$ . Then  $(Z, W, \varphi; g; f) \in \text{Fr}_m^{qf}(U, Y/(Y-S))$  if and only if for  $i, j = 1, 2$  and  $j \neq i$  one has

$$(Z_i, W \setminus Z_j, \varphi|_{W \setminus Z_j}; g|_{W \setminus Z_j}; f|_{W \setminus Z_j}) \in \text{Fr}_m^{qf}(U, Y/(Y-S)).$$

This observation leads to the following

**Definition 3.1.** Let  $Y \in \text{Sm}/k$  and  $S \subset Y$  be as in Definition 2.1. Let  $U$  be a  $k$ -smooth scheme. Set

$$\mathbb{Z}\text{F}_m^{qf}(U, Y/(Y-S) \wedge T) := \mathbb{Z}[\text{Fr}_m^{qf}(U, Y/(Y-S) \wedge T)]/A,$$

where  $A$  is the subgroup generated by the elements

$$(Z \sqcup Z', W, \varphi; g; f) - (Z, W \setminus Z', \varphi|_{W \setminus Z'}; g|_{W \setminus Z'}; f|_{W \setminus Z'}) - (Z', W \setminus Z, \varphi|_{W \setminus Z}; g|_{W \setminus Z}; f|_{W \setminus Z}).$$

Set  $\mathbb{Z}\text{F}_*^{qf}(U, Y/(Y-S) \wedge T) = \bigoplus_{m \geq 0} \mathbb{Z}\text{F}_m^{qf}(U, Y/(Y-S) \wedge T)$ . The pairing (5) gives rise to a natural pairing  $\mathbb{Z}\text{F}_k(V, U) \times \mathbb{Z}\text{F}_m^{qf}(U, Y/(Y-S)) \rightarrow \mathbb{Z}\text{F}_{k+m}^{qf}(U, Y/(Y-S))$ . The latter pairing makes  $\mathbb{Z}\text{F}_*^{qf}(-, Y/(Y-S))$  a linear framed presheaf.

The external product (6) gives rise to an external product

$$\mathbb{Z}\text{F}_m^{qf}(U, Y/(Y-S)) \wedge T \times \mathbb{Z}\text{F}_n(\text{pt}, \text{pt}) \xrightarrow{-\boxtimes -} \mathbb{Z}\text{F}_{m+n}^{qf}(U, Y/(Y-S) \wedge T)$$

**Definition 3.2.** Set,

$$\mathbb{Z}\text{F}^{qf}(-, Y/(Y-S) \wedge T) = \text{colim}(\mathbb{Z}\text{F}_0^{qf}(-, Y/(Y-S) \wedge T) \xrightarrow{\Sigma} \mathbb{Z}\text{F}_1^{qf}(-, Y/(Y-S) \wedge T) \xrightarrow{\Sigma} \dots)$$

By the very construction  $\mathbb{Z}\text{F}^{qf}(-, Y/(Y-S) \wedge T)$  is a linear framed presheaf.



**Definition 3.3.** For  $U \in Sm/k$  let  $F_m(U, Y/(Y - S) \wedge T)$  be a subset of  $Fr_m(U, Y/(Y - S) \wedge T)$  consisting of elements

$$(Z, W, \phi_1, \dots, \phi_m; g; f) \in Fr_m(U, Y/(Y - S) \wedge T)$$

such that  $Z$  is connected. Clearly, this definition is consistent with Definition 2.4. Set,

$$F_m^{qf}(U, Y/(Y - S) \wedge T) = F_m(U, Y/(Y - S) \wedge T) \cap Fr_m^{qf}(U, Y/(Y - S) \wedge T).$$

$F_m(U, Y/(Y - S) \wedge T) \setminus *$  is plainly a free basis of the free abelian group  $\mathbb{Z}F_m(U, Y/(Y - S) \wedge T)$ .

The following lemma is obvious.

**Lemma 3.4.** (1) *The set  $(F_m^{qf}(U, Y/(Y - S) \wedge T) \setminus *)$  is a free basis of  $\mathbb{Z}F_m^{qf}(U, Y/(Y - S) \wedge T)$ .*

(2) *The natural map  $\mathbb{Z}F_m^{qf}(U, Y/(Y - S) \wedge T) \rightarrow \mathbb{Z}F_m(U, Y/(Y - S) \wedge T)$  is injective and identifies  $\mathbb{Z}F_m^{qf}(U, Y/(Y - S) \wedge T)$  with a direct summand of  $\mathbb{Z}F_m(U, Y/(Y - S) \wedge T)$ . Therefore  $\mathbb{Z}F_m^{qf}(-, Y/(Y - S) \wedge T)$  is a framed subpresheaf of the framed presheaf  $\mathbb{Z}F(-, Y/(Y - S) \wedge T)$ .*

#### 4. USEFUL LEMMAS

In this section we prove a couple of useful lemmas used in the following sections. We start with a useful remark.

**Remark 4.1.** Let  $W$  be a local regular scheme with a closed point  $w$ . Let  $S \subset W$  be a closed subset. Then  $S$  is a local scheme which is connected and  $w$  is the only closed point of  $S$ .

Furthermore, let  $U$  be a henselian local regular scheme with the closed point  $u$ . Let  $\pi : W \rightarrow U$  be a morphism such that  $\pi(w) = u$  and  $\pi|_S : S \rightarrow U$  is quasi-finite. By [Mi, Theorem I.4.2]  $S$  is finite over  $U$ . Regarding  $S$  as a closed subscheme with the reduced structure, it is connected and local.

Let  $S$  be a henselian local scheme and let  $Z$  be its closed subset. Let  $S_Z^h$  be the henselization of  $S$  at  $Z$ . Then the canonical morphism  $can_{S,Z} : S_Z^h \rightarrow S$  is an isomorphism. Indeed,  $(S, id_S, i : Z \hookrightarrow S)$  is the initial object in the category of étale neighborhoods of  $Z$  in  $S$ .

**Lemma 4.2.** *Let  $V$  be an affine scheme,  $Z \subset V$  be a closed connected subset,  $can = can_{V,Z} : V_Z^h \rightarrow V$  be the henselization of  $V$  at  $Z$ , and let  $s : Z \rightarrow V_Z^h$  be the section of  $can$  over  $Z$ . Let  $U$  be a regular local henselian scheme,  $q : V \rightarrow U$  be a smooth morphism such that the morphism  $q|_Z : Z \rightarrow U$  is finite. Furthermore, suppose  $Y \subset V_Z^h$  is a closed subset containing  $s(Z)$ , which is quasi-finite over  $U$ . Then  $can|_Y : Y \rightarrow V$  is a closed embedding,  $can(Y)$  contains  $Z$  and  $can^{-1}(can(Y)) = Y$ .*

*Proof.* Since  $Z$  is finite over the local henselian  $U$  and  $Z$  is connected, it is local and henselian. Since  $Z$  is local, then so is the scheme  $W = V_Z^h$ . By Remark 4.1 the scheme  $Y$  is local, connected and finite over  $U$ . Hence  $Y_1 = can(Y)$  is closed in  $V$  and finite over  $U$  and contains  $Z$ . So,  $Y_1$  is a local henselian scheme.

The scheme  $can^{-1}(Y_1)$  is the henselization of  $Y_1$  at  $Z$ . By Remark 4.1 the morphism  $p_1 = can|_{can^{-1}(Y_1)} : can^{-1}(Y_1) \rightarrow Y_1$  is an isomorphism. Clearly,  $Y$  is a closed subset of  $can^{-1}(Y_1)$  and let  $i : Y \rightarrow can^{-1}(Y_1)$  be the inclusion. Then the ring map  $(p_1 \circ i)^* : \Gamma(Y_1, \mathcal{O}_{Y_1}) \rightarrow \Gamma(Y, \mathcal{O}_Y)$  is surjective. On the other hand, this ring morphism is injective, because both schemes are reduced affine and the morphism  $p_1 \circ i$  is surjective. Hence  $p_1 \circ i : Y \rightarrow Y_1$  is an isomorphism. Thus  $can|_Y : Y \rightarrow V$  is a closed embedding.  $\square$

**Lemma 4.3.** *Under the assumptions of Lemma 4.2 let  $Y \subset V$  be a closed connected subset containing  $Z$  and be finite over  $U$ . Then there is a unique section  $t : Y \rightarrow V_Z^h$  of the morphism  $can : V_Z^h \rightarrow V$  and  $t(Y) = can^{-1}(Y)$  contains  $s(Z)$ .*

*Proof.* Since  $U$  is local henzelian, then so is the scheme  $Y$ . The scheme  $\text{can}^{-1}(Y)$  is the henzelization of  $Y$  at  $Z$ . By Remark 4.1 the morphism  $p = \text{can}|_{\text{can}^{-1}(Y)} : \text{can}^{-1}(Y) \rightarrow Y$  is an isomorphism. Set  $t = \text{in} \circ p^{-1} : Y \rightarrow V_Z^h$ . Clearly,  $t$  is a section of  $\text{can}$  over  $Y$ . If  $t' : Y \rightarrow V_Z^h$  is another section of  $\text{can}$ , then  $t' = t$ .  $\square$

## 5. THE LINEAR FRAMED PRESHEAF $\mathbb{Z}\mathbf{F}^{qf}(X_+ \wedge T^{n+1})$

Let  $X$  be a  $k$ -smooth variety and let  $Y = X \times \mathbb{A}^n$  and  $S = X \times \{0\}$  one has an equality of the Nisnevich sheaves  $Y/(Y-S) = X_+ \wedge T^n$ . Thus there are framed presheaves

$$\text{Fr}^{qf}(-, X_+ \wedge T^{n+1}) := \text{Fr}^{qf}(-, Y/(Y-S) \wedge T) \quad \text{and} \quad \text{Fr}(-, X_+ \wedge T^{n+1}) := \text{Fr}(-, Y/(Y-S) \wedge T),$$

$$\mathbb{Z}\mathbf{F}^{qf}(-, X_+ \wedge T^{n+1}) := \mathbb{Z}\mathbf{F}^{qf}(-, Y/(Y-S) \wedge T) \quad \text{and} \quad \mathbb{Z}\mathbf{F}(-, X_+ \wedge T^{n+1}) := \mathbb{Z}\mathbf{F}(-, Y/(Y-S) \wedge T).$$

There are also framed presheaves:

- $\diamond \text{Fr}_*^{qf}(-, X_+ \wedge T^{n+1}) := \text{Fr}_*^{qf}(-, Y/(Y-S) \wedge T),$
- $\diamond \text{Fr}_*(-, X_+ \wedge T^{n+1}) := \text{Fr}_*(-, Y/(Y-S) \wedge T),$
- $\diamond \mathbb{Z}\mathbf{F}_*^{qf}(-, X_+ \wedge T^{n+1}) := \mathbb{Z}\mathbf{F}_*^{qf}(-, Y/(Y-S) \wedge T),$
- $\diamond \mathbb{Z}\mathbf{F}_*(-, X_+ \wedge T^{n+1}) := \mathbb{Z}\mathbf{F}_*(-, Y/(Y-S) \wedge T),$

and sets

- $\diamond \mathbf{F}_m(U, X_+ \wedge T^{n+1}) := \mathbf{F}_m(U, Y/(Y-S) \wedge T),$
- $\diamond \mathbf{F}_m^{qf}(U, X_+ \wedge T^{n+1}) := \mathbf{F}_m^{qf}(U, Y/(Y-S) \wedge T).$

Specifying definitions of Section 3, a section of  $\text{Fr}_m(-, X_+ \wedge T^{n+1})$  on  $U \in \text{Sm}/k$  is a tuple

$$c = (Z, W, \varphi_1, \dots, \varphi_m; h : W \rightarrow X \times \mathbb{A}^1; f : W \rightarrow \mathbb{A}^1),$$

where  $Z$  is a closed subset of  $U \times \mathbb{A}^m$  finite over  $U$ ,  $W$  is an étale neighborhood of  $Z$  in  $\mathbb{A}^m \times U$ ,  $\varphi_1, \dots, \varphi_m, f$  are regular functions on  $W$ ,  $h = (g, \varphi_{m+1}, \dots, \varphi_{m+n}) : W \rightarrow X \times \mathbb{A}^n$  is a regular map such that

$$Z = Z(\varphi_1, \dots, \varphi_m, f) \cap h^{-1}(X \times \{0\}) = Z(\varphi_1, \dots, \varphi_m, f, \varphi_{m+1}, \dots, \varphi_{m+n}).$$

The section  $c$  is in  $\text{Fr}_m^{qf}(U, X_+ \wedge T^{n+1})$  if and only if the vanishing locus  $Z(\varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n})$  is *quasi-finite* over  $U$ .

The section  $c$  is in  $\mathbf{F}_m(U, X_+ \wedge T^{n+1})$  if and only if the set  $Z$  is *connected*. The section  $c$  is in  $\mathbf{F}_m^{qf}(U, X_+ \wedge T^{n+1})$  if and only if the set  $Z$  is *connected* and the vanishing locus  $Z(\varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n})$  is *quasi-finite* over  $U$ .

The suspension map  $\Sigma : \text{Fr}_m(-, X_+ \wedge T^{n+1}) \rightarrow \text{Fr}_{m+1}(-, X_+ \wedge T^{n+1})$  sends  $(Z, W, \varphi_1, \dots, \varphi_m; g; \varphi_{m+1}, \dots, \varphi_{m+n}; f)$  to  $(Z \times \{0\}, W \times \mathbb{A}^1, \varphi_1, \dots, \varphi_m, t; g; \varphi_{m+1}, \dots, \varphi_{m+n}; f)$ .

**Notation 5.1.** For the convenience of computations we shall write  $(Z, W, \varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n}; f : W \rightarrow \mathbb{A}^1, g : W \rightarrow X)$  for  $(Z, W, \varphi_1, \dots, \varphi_m; g : W \rightarrow X; \varphi_{m+1}, \dots, \varphi_{m+n}; f : W \rightarrow \mathbb{A}^1)$  in the rest of the paper.

The canonical morphism  $\mathbb{A}^1 \rightarrow T$  induces a morphism of framed presheaves

$$p : \text{Fr}_*(-, X_+ \wedge T^n \wedge \mathbb{A}_+^1) \rightarrow \text{Fr}_*(-, X_+ \wedge T^{n+1}),$$

sending  $(Z, W, \varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n}; f; g)$  to  $(Z', W, \varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n}, \varphi_{m+n+1}; g)$ , where  $Z' = Z \cap Z(f)$ ,  $\varphi_{m+n+1} = f$ .



**Lemma 5.2.** *If  $U$  is essentially  $k$ -smooth local henzelian, then the image of the map  $p : \text{Fr}_*(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1) \rightarrow \text{Fr}_*(U, X_+ \wedge T^{n+1})$  is contained in  $\text{Fr}_*^{qf}(U, X_+ \wedge T^{n+1})$ . Moreover, for any  $m \geq 0$  the map  $p$  sends  $F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1)$  to the set  $F_m^{qf}(U, X_+ \wedge T^{n+1})$ . Finally, the square of pointed sets*

$$\begin{array}{ccc} F_m(U, X_+ \wedge T^n \wedge \mathbb{G}_{m,+}) & \xrightarrow{i} & F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1) \\ \downarrow & & \downarrow p \\ * & \longrightarrow & F_m^{qf}(U, X_+ \wedge T^{n+1}), \end{array}$$

is a pushout square. Here  $* \in F_m(U, X_+ \wedge T^{n+1})$  is the empty framed correspondence.

*Proof.* The first assertion is obvious. To prove the second one, take an element

$$c = (Y, W, \varphi_1, \dots, \varphi_{m+n}, f : W \rightarrow \mathbb{A}^1; g : W \rightarrow X) \in F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1)$$

with  $W = (\mathbb{A}^m \times U)_Y^h$ . Then  $p(c) = (Z, W, \varphi_1, \dots, \varphi_{m+n+1}; g)$ , where  $Z = Y \cap Z(f)$ ,  $\varphi_{m+n+1} = f$ .

Since  $Y$  is connected and finite over the henzelian  $U$ , then  $Y$  is henzelian and local. Hence  $W$  is local. By Remark 4.1 the closed subset  $Z$  is connected, whence the second assertion of the lemma.

To prove the third assertion, it is sufficient to construct a section

$$s : F_m^{qf}(U, X_+ \wedge T^{n+1}) \setminus * \rightarrow F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1) \setminus F_m(U, X_+ \wedge T^n \wedge \mathbb{G}_{m,+})$$

of  $p$  and check that the map  $p$  is injective on the complement of  $F_m(U, X_+ \wedge T^n \wedge \mathbb{G}_{m,+})$ .

We construct  $s$  as follows. Take an element  $c = (Z, \mathbb{A}^m \times U \xleftarrow{\text{can}} W, \varphi_1, \dots, \varphi_{m+n+1}; g)$  in  $F_m^{qf}(U, X_+ \wedge T^{n+1})$  with a non-empty  $Z$  and with  $W = (\mathbb{A}^m \times U)_Z^h$ . Since  $Z$  is connected and finite over the local henzelian  $U$ , the scheme  $W$  is local. Set

$$s(c) = (\text{can}(Y), \mathbb{A}^m \times U \xleftarrow{\text{can}} W, \varphi_1, \dots, \varphi_{m+n}; \varphi_{m+n+1} : W \rightarrow \mathbb{A}^1; g : W \rightarrow X),$$

where  $Y = Z(\varphi_1, \dots, \varphi_{m+n}) \subset W$ . The set  $Y$  is quasi-finite over  $U$ , because  $c \in F_m^{qf}(U, X_+ \wedge T^{n+1})$ . By Remark 4.1 the set  $Y$  is finite over  $U$  and connected and local. By Lemma 4.2 the morphism  $\text{can}|_Y : Y \rightarrow \mathbb{A}^m \times U$  is a closed embedding and  $\text{can}^{-1}(\text{can}(Y)) = Y$ . Thus  $s(c) \in F_m(U, X_+ \wedge T^{n+1} \wedge \mathbb{A}_+^1)$ . Clearly,  $p(s(c)) = c$ .

Now check the required injectivity for  $p$ . Let  $c' = (Y', W', \varphi'_1, \dots, \varphi'_{m+n}, f' : W' \rightarrow \mathbb{A}^1; g' : W' \rightarrow X)$  and  $c'' = (Y'', W'', \varphi''_1, \dots, \varphi''_{m+n}, f'' : W'' \rightarrow \mathbb{A}^1; g'' : W'' \rightarrow X)$  in  $F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1)$  be two elements with  $W' = (\mathbb{A}^m \times U)_{Y'}^h$  and  $W'' = (\mathbb{A}^m \times U)_{Y''}^h$ . Let  $\text{can}' : W' \rightarrow \mathbb{A}^m \times U$  be the canonical morphism and let  $s' : Y' \rightarrow W'$  be the section of  $\text{can}'$ .

Suppose that  $p(c) = p(c')$  and the support  $Z = Y' \cap Z(f') = Y'' \cap Z(f'')$  is non-empty. We must check that  $c' = c''$ . The element  $p(c')$  is of the form

$$(Z, \mathbb{A}^m \times U \xleftarrow{\text{can}} W, \varphi_1, \dots, \varphi_{m+n+1}; g),$$

where  $Z = Y' \cap Z(f')$ ,  $W = (\mathbb{A}^m \times U)_Z^h$ ,  $\varphi_i = \varphi_i|_W$ ,  $g = g|_W$ . Consider the canonical morphism  $\text{can}_1 : W \rightarrow W'$ . It is the henzelization of  $W'$  at  $s'(Y')$ . Note that  $s'(Y') \subset W'$  is a closed subset containing  $s'(Z)$ . Moreover,  $Y'$  is finite over the henzelian  $U$  and connected. By Lemma 4.3 there is a unique section  $t' : Y' \rightarrow W$  of the morphism  $\text{can}_1$ , and  $t'(Y') = \text{can}_1^{-1}(Y')$  contains  $s(Z)$ , where  $s : Z \rightarrow W$  is the section of  $\text{can}_1$  (the morphism  $s$  is also the section of  $\text{can} = \text{can}' \circ \text{can}_1$ ). These arguments imply an equality

$$c' = (Y', W, \varphi'_1|_W, \dots, \varphi'_{m+n}|_W, f'|_W : W \rightarrow \mathbb{A}^1; g'|_W : W \rightarrow X) \in F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1).$$

By the same reason one has an equality

$$c'' = (Y'', W, \varphi''_1|_W, \dots, \varphi''_{m+n}|_W, f''|_W : W \rightarrow \mathbb{A}^1; g''|_W : W \rightarrow X) \in F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1).$$

Since  $W$  is the henselization of  $\mathbb{A}^m \times U$  at  $Z$  and  $p(c') = p(c'')$ , one has equalities  $\phi'_i|_W = \phi''_i|_W$  for  $i = 1, \dots, m+n$ ,  $f'|_W = f''|_W$  and  $g'|_W = g''|_W$ . Hence  $Y' = Y''$  and, moreover,  $c' = c''$  in  $F_m(U, X_+ \wedge T^n \wedge \mathbb{A}_+^1)$ . The desired injectivity is proved. The section  $s$  is constructed above and our lemma follows.  $\square$

**Corollary 5.3.** *For any integer  $n \geq 0$  the natural morphism*

$$\alpha_* : \mathbb{Z}F_*(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / \mathbb{Z}F_*(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow \mathbb{Z}F_*^{qf}(X_+ \wedge T^{n+1})$$

*of  $\mathbb{Z}F_*(k)$ -presheaves is an isomorphism locally for the Nisnevich topology. As a consequence, the natural morphism*

$$\alpha : \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow \mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$$

*of  $\mathbb{Z}F_*(k)$ -presheaves is an isomorphism locally for the Nisnevich topology.*

We are now in a position to prove Theorem 1.4.

*Proof of Theorem 1.4.* The theorem is implied by Definitions 3.2, 3.3 and Corollary 5.3.  $\square$

**Lemma 5.4.** *Let  $k$  be an infinite perfect field. Let  $A$  and  $B$  be linear framed presheaves such that the cohomology presheaves of the complexes  $C_*(A)$  and  $C_*(B)$  are quasi-stable (see [GP1, Def. 2.6] for the definition of quasi-stability). Let  $\alpha : A \rightarrow B$  be a morphism of linear framed presheaves, which is an isomorphism locally in the Nisnevich topology. Then the morphism*

$$C_*(\alpha) : C_*(A) \rightarrow C_*(B)$$

*is a quasi-isomorphism locally in the Nisnevich topology.*

*Proof.* By assumption the map of the Eilenberg–Mac Lane spectra  $\alpha : EM(A) \rightarrow EM(B)$  is a local weak equivalence. Hence the induced map  $\alpha : EM(C_*(A)) \rightarrow EM(C_*(B))$  is a motivic weak equivalence of  $S^1$ -spectra. By assumption the stable homotopy groups of the spectra  $EM(C_*(A)), EM(C_*(B))$  are radditive, quasi-stable and  $\mathbb{A}^1$ -invariant (see [GP2, Introduction] for the definition of radditivity). The Resolution Theorem of [GP1] implies  $\alpha : EM(C_*(A)) \rightarrow EM(C_*(B))$  is a Nisnevich local weak equivalence of  $S^1$ -spectra. Thus the morphism of complexes  $C_*(\alpha) : C_*(A) \rightarrow C_*(B)$  is a quasi-isomorphism locally in the Nisnevich topology.  $\square$

We are now in a position to prove a statement which is necessary for the proof of Theorem 1.3.

**Proposition 5.5.** *Let  $k$  be an infinite perfect field. Then the morphism*

$$C_*(\alpha) : C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow C_*\mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$$

*is a quasi-isomorphism locally in the Nisnevich topology.*

*Proof.* The morphism  $\alpha : \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow \mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$  is a morphism of linear framed presheaves. Set  $A = \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / \mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+})$  and  $B = \mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$ . Then the cohomology presheaves of the complexes  $C_*(A)$  and  $C_*(B)$  are quasi-stable by the construction of  $A$  and  $B$ . Now Theorem 1.4 and Lemma 5.4 imply the claim.  $\square$

Thus we have computed locally the complex  $C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1) / C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+})$  as the complex  $C_*\mathbb{Z}F^{qf}(X_+ \wedge T^{n+1})$ . Our next goal is to show that the latter complex is quasi-isomorphic to  $C_*\mathbb{Z}F(X_+ \wedge T^{n+1})$  locally in the Zarisky topology verifying Theorem 1.5. The next two sections are dedicated to this theorem.

## 6. A FILTRATION ON $\mathbb{Z}F_n(-, X_+ \wedge T^{n+1})$

We start with the following

**Definition 6.1.** Let  $U \in Sm/k$  be an affine variety and let  $c = (Z, W, \varphi; g) \in \text{Fr}_m(U, X_+ \wedge T^{n+1})$  be a framed correspondence. A finite set of polynomials  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times U]$  is said to be *c-defining* if for every point  $u \in U$  there is  $i \in \{1, 2, \dots, r\}$  such that

- ◊ the polynomial  $F_i(-, u) \in k(u)[\mathbb{A}^{m+n+1}]$  is nonzero,
- ◊  $\varphi(W_u) \subseteq Z(F_i(-, u))$  in  $\mathbb{A}_u^{m+n+1}$ .

Note that if a finite set of polynomials  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times U]$  is *c-defining*, where  $c = (Z, W, \varphi; g) = (Z_1 \sqcup Z_2, W, \varphi; g)$ , then the same collection of polynomials is  $(Z_1, W - Z_2, \varphi; g)$ -defining and is  $(Z_2, W - Z_1, \varphi; g)$ -defining respectively.

The following lemma is crucial in our analysis.

**Lemma 6.2.** Let  $m, n \geq 0$  and  $Y$  an affine (possibly non-irreducible)  $k$ -variety. Let  $W \rightarrow \mathbb{A}^m \times Y$  be an étale morphism and  $\psi : W \rightarrow \mathbb{A}^{m+n+1} \times Y$  be a morphism of  $Y$ -schemes. Then there is a finite set of polynomials  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times Y]$  such that for every point  $y \in Y$  there is  $i \in \{1, 2, \dots, r\}$  such that

- ◊ the polynomial  $F_i(-, y) \in k(y)[\mathbb{A}^{m+n+1}]$  is nonzero and
- ◊  $\varphi(W_y) \subseteq Z(F_i(-, y))$  in  $\mathbb{A}_y^{m+n+1}$ .

*Proof.* We proceed by induction in the dimension of  $Y$ . If  $\dim(Y) = 0$  then there is nothing to prove. Now suppose  $\dim(Y) > 0$ . Let  $Y_1, \dots, Y_l$  be all irreducible components of  $Y$ . For an index  $i$  from  $\{1, \dots, l\}$  take the restriction of the map  $\psi|_{W_{Y_i}} : W_{Y_i} \rightarrow \mathbb{A}^{m+n+1} \times Y_i$  to  $Y_i$ . The dimension of  $W_{Y_i}$  is  $m + \dim(Y_i)$ . Thus the closure of its image is contained in the zero locus  $Z(\bar{F}_{(i)})$  of a non-zero polynomial  $\bar{F}_{(i)} \in k[\mathbb{A}^{m+n+1} \times Y_i]$ . Since  $Y_i$  is closed in the affine variety  $Y$ , the polynomial  $\bar{F}_i$  can be extended to a polynomial  $F_i \in k[\mathbb{A}^{m+n+1} \times Y]$ . Let  $V \subset Y$  be an open subset consisting of those  $y \in Y$  such that there is  $i$  with  $0 \neq F_i(-, y)$  in  $k(y)[\mathbb{A}^{m+n+1}]$ . Let  $Y' = Y - V$ .

By construction,  $V$  has a non-empty intersection with every irreducible component of  $Y$ . Thus  $\dim(Y') < \dim(Y)$ . By the inductive assumption there are polynomials  $\bar{F}_{l+1}, \dots, \bar{F}_r$  in  $k[\mathbb{A}^{m+n+1} \times Y']$  such that for every point  $y \in Y'$  there is  $j \in \{l+1, \dots, r\}$  with each polynomial  $\bar{F}_j(-, y) \in k(y)[\mathbb{A}^{m+n+1}]$  nonzero and  $\psi(W_y) \subseteq Z(\bar{F}_j(-, y))$  in  $\mathbb{A}_y^{m+n+1}$ . Since  $Y'$  is closed in the affine  $Y$  for any  $j$ , the polynomial  $\bar{F}_j$  can be extended to a polynomial  $F_j$  in  $k[\mathbb{A}^{m+n+1} \times Y]$ . Clearly, the set of polynomials  $F_i$ , where  $i \in \{1, 2, \dots, r\}$ , are the desired polynomials for  $Y$ .  $\square$

The preceding lemma has the following

**Corollary 6.3.** Let  $U \in Sm/k$  be an affine variety and let  $c = (Z, W, \varphi; g) \in \text{Fr}_m(U, X_+ \wedge T^{n+1})$  be a framed correspondence. Then there exists a *c-defining* set of polynomials  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times U]$ .

Moreover, if  $f : V \rightarrow U$  is a morphism of  $k$ -smooth affine varieties and  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times U]$  is a *c-defining* set, then  $f^*(F_1), \dots, f^*(F_r) \in k[\mathbb{A}^{m+n+1} \times V]$  is a  $f^*(c)$ -defining set.

Let  $U \in Sm/k$  be an affine variety. Let  $d > 0$ . Define  $\text{Fr}_m^{<d}(U, X_+ \wedge T^{n+1})$  as a subset of  $\text{Fr}_m(U, X_+ \wedge T^{n+1})$  consisting of those  $c = (Z, W, \varphi; g) \in \text{Fr}_m(U, X_+ \wedge T^{n+1})$  for which there exists a *c-defining* set  $F_1, \dots, F_r \in k[\mathbb{A}^{m+n+1} \times U]$  with  $\deg F_i < d$  for all  $i = 1, \dots, r$ . Set,

$$(\text{Fr}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1}) := \text{Fr}_m^{qf}(U, X_+ \wedge T^{n+1}) \cap \text{Fr}_m^{<d}(U, X_+ \wedge T^{n+1}),$$

Corollary 6.3 has the following obvious application.

**Lemma 6.4.** *For any integers  $m, n \geq 0$  and any integer  $d > 0$  the following statements are true:*

- (i)  $\mathrm{Fr}_m^{<d}(-, X_+ \wedge T^{n+1})$  is a subpresheaf on  $\mathrm{AffSm}/k$  of the presheaf  $\mathrm{Fr}_m(-, X_+ \wedge T^{n+1})$ ;
- (ii) the increasing filtration of the presheaf  $\mathrm{Fr}_m(-, X_+ \wedge T^{n+1})|_{\mathrm{AffSm}/k}$  by subpresheaves  $\mathrm{Fr}_m^{<d}(-, X_+ \wedge T^{n+1})$  is exhausting;
- (iii)  $(\mathrm{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1})$  is a subpresheaf on  $\mathrm{AffSm}/k$  of the presheaf  $\mathrm{Fr}_m^{qf}(-, X_+ \wedge T^{n+1})$ ;
- (iv) the increasing filtration of the presheaf  $\mathrm{Fr}_m^{qf}(-, X_+ \wedge T^{n+1})|_{\mathrm{AffSm}/k}$  by subpresheaves  $(\mathrm{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1})$  is exhausting.

**Definition 6.5.** For an affine  $k$ -smooth  $U$  we define  $\mathbb{Z}\mathrm{F}_m^{<d}(U, X_+ \wedge T^{n+1})$  as

$$\mathbb{Z}[\mathrm{Fr}_m^{<d}(U, X_+ \wedge T^{n+1})] / \langle (Z_1 \sqcup Z_2, W, \varphi; g) - (Z_1, W_2, \varphi|_{W_2}; g|_{W_2}) - (Z_2, W_1, \varphi|_{W_1}; g|_{W_1}) \rangle,$$

where  $W_i = W - Z_i$  for  $i = 1, 2$ . By Lemma 6.4 the assignment  $U \mapsto \mathbb{Z}\mathrm{F}_m^{<d}(U, X_+ \wedge T^{n+1})$  is a presheaf on  $\mathrm{AffSm}/k$ .

Likewise, for an affine  $k$ -smooth  $U$  define  $(\mathbb{Z}\mathrm{F}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1})$  as

$$\mathbb{Z}[(\mathrm{Fr}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1})] / \langle (Z_1 \sqcup Z_2, W, \varphi; g) - (Z_1, W_2, \varphi|_{W_2}; g|_{W_2}) - (Z_2, W_1, \varphi|_{W_1}; g|_{W_1}) \rangle,$$

where  $W_i = W - Z_i$  for  $i = 1, 2$ . By Lemma 6.4 the assignment  $U \mapsto (\mathbb{Z}\mathrm{F}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1})$  is a presheaf on  $\mathrm{AffSm}/k$ .

Recall that  $\mathbb{Z}\mathrm{F}_m(-, X_+ \wedge T^{n+1})$  is a presheaf on  $\mathrm{Sm}/k$  (see Definition 2.3) given by

$$\mathbb{Z}[\mathrm{Fr}_m(U, X_+ \wedge T^{n+1})] / \langle (Z_1 \sqcup Z_2, W, \varphi; g) - (Z_1, W_2, \varphi|_{W_2}; g|_{W_2}) - (Z_2, W_1, \varphi|_{W_1}; g|_{W_1}) \rangle,$$

where  $W_i = W - Z_i$  for  $i = 1, 2$ . In turn,  $\mathbb{Z}\mathrm{F}_m^{qf}(-, X_+ \wedge T^{n+1})$  is a presheaf on  $\mathrm{Sm}/k$  (see Definition 3.1) given by

$$\mathbb{Z}[\mathrm{Fr}_m^{qf}(U, X_+ \wedge T^{n+1})] / \langle (Z_1 \sqcup Z_2, W, \varphi; g) - (Z_1, W_2, \varphi|_{W_2}; g|_{W_2}) - (Z_2, W_1, \varphi|_{W_1}; g|_{W_1}) \rangle,$$

where  $W_i = W - Z_i$  for  $i = 1, 2$ .

For any positive integers  $d < d'$  the inclusion  $\mathrm{Fr}_m^{<d}(-, X_+ \wedge T^{n+1}) \subset \mathrm{Fr}_m^{<d'}(-, X_+ \wedge T^{n+1})$  induces a morphism of Abelian presheaves on  $\mathrm{AffSm}/k$

$$\mathbb{Z}\mathrm{F}_m^{<d}(U, X_+ \wedge T^{n+1}) \rightarrow \mathbb{Z}\mathrm{F}_m^{<d'}(U, X_+ \wedge T^{n+1}).$$

Likewise, for any positive  $d < d'$  the inclusion  $(\mathrm{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) \subset (\mathrm{Fr}_m^{qf})^{<d'}(-, X_+ \wedge T^{n+1})$  induces a morphism of Abelian presheaves on  $\mathrm{AffSm}/k$

$$(\mathbb{Z}\mathrm{F}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1}) \rightarrow (\mathbb{Z}\mathrm{F}_m^{qf})^{<d'}(U, X_+ \wedge T^{n+1}).$$

The next statement is a consequence of Lemma 6.4.

**Corollary 6.6.** *One has two equalities of presheaves on  $\mathrm{AffSm}/k$*

$$\mathrm{colim}_d \mathbb{Z}\mathrm{F}_m^{<d}(-, X_+ \wedge T^{n+1}) = \mathbb{Z}\mathrm{F}_m(-, X_+ \wedge T^{n+1})$$

and

$$\mathrm{colim}_d (\mathbb{Z}\mathrm{F}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) = \mathbb{Z}\mathrm{F}_m^{qf}(-, X_+ \wedge T^{n+1}).$$

*Proof.* This is straightforward. □

## 7. MOVING LEMMA

**Lemma 7.1.**<sup>1</sup> *Let  $L$  be a field and  $F \in L[x_1, \dots, x_{r+1}]$  a nonzero polynomial such that  $\deg F \leq (d-1)$ . Then the polynomials  $F(t^{d^r}, t^{d^{r-1}}, \dots, t^d, t)$  and  $F(t^d, \dots, t^{d^{r-1}}, t^{d^r}, t)$  are both non-zero in  $L[t]$ . Moreover, for any non-zero  $s \in L$  the polynomials  $F(st^{d^r}, st^{d^{r-1}}, \dots, st^d, t)$  and  $F(st^d, \dots, st^{d^{r-1}}, st^{d^r}, t)$  are both non-zero.*

*Proof.* Let us prove that the first polynomial is non-zero. If  $F = \sum_{I=(i_1, \dots, i_{r+1})} a_I x^I$  then

$$F(t^{d^r}, t^{d^{r-1}}, \dots, t^d, t) = \sum a_I t^{i_1 d^r + \dots + i_r d + i_{r+1}} \quad (7)$$

Let us check that if  $I$  and  $J$  are two different multi-indices, then  $I \cdot (d^r, \dots, d, 1) \neq J \cdot (d^r, \dots, d, 1)$ . Indeed, if these are equal, then

$$d^r(i_1 - j_1) + \dots + d(i_n - j_n) + (i_{n+1} - j_{n+1}) = 0.$$

It follows that  $i_{n+1} - j_{n+1}$  is divisible by  $d$ , but  $|i_{n+1} - j_{n+1}| \leq d-1$ , hence  $i_{n+1} = j_{n+1}$ . Then  $(i_n - j_n)$  is divisible by  $d$ , hence zero and  $(i_1 - j_1)$  is zero by induction. Thus all powers of  $t$  in the sum (7) are distinct. So if some  $a_I \neq 0$  then the right hand side of (7) is nonzero. The second polynomial is obtained from the first one by permuting powers of  $t$ . Thus it is non-zero. For any multi-index  $I$  the coefficient at  $t^{i_1 d^r + \dots + i_r d + i_{r+1}}$  in the polynomial  $F(st^{d^r}, st^{d^{r-1}}, \dots, st^d, t)$  is obtained from  $a_I$  by multiplying a power of  $s$ . We see that the polynomial  $F(st^{d^r}, st^{d^{r-1}}, \dots, st^d, t)$  is non-zero, and hence so is  $F(st^d, \dots, st^{d^{r-1}}, st^{d^r}, t)$ .  $\square$

Let  $U \in \text{Sm}/k$  be an affine variety and let  $c = (Z, W, \varphi) \in \text{Fr}_m(U, X_+ \wedge T^{n+1}; g)$  be a framed correspondence. Let  $d > 0$  be an integer. Set

$$t_d(c) = (Z, W, \varphi_1 - \varphi_{m+n+1}^d, \varphi_2 - \varphi_{m+n+1}^{d^2}, \dots, \varphi_{m+n} - \varphi_{m+n+1}^{d^{m+n}}, \varphi_{m+n+1}; g).$$

Note that  $Z(\varphi_1 - \varphi_{m+n+1}^d, \varphi_2 - \varphi_{m+n+1}^{d^2}, \dots, \varphi_{m+n} - \varphi_{m+n+1}^{d^{m+n}}, \varphi_{m+n+1}) = Z(\varphi_1, \dots, \varphi_{m+n}, \varphi_{m+n+1})$  in  $W$ . Thus the tuple  $t_d(c)$  is an element in  $\text{Fr}_n(U, X_+ \wedge T^{n+1})$  such that its support  $Z$  is the same with that of  $c$ . If  $s$  is a variable, we put

$$h^d(c) = (Z \times \mathbb{A}^1, W \times \mathbb{A}^1, \varphi_1 - s\varphi_{m+n+1}^d, \varphi_2 - s\varphi_{m+n+1}^{d^2}, \dots, \varphi_{m+n} - s\varphi_{m+n+1}^{d^{m+n}}, \varphi_{m+n+1}; g).$$

Note that  $Z(\varphi_1 - s\varphi_{m+n+1}^d, \dots, \varphi_{m+n} - s\varphi_{m+n+1}^{d^{m+n}}, \varphi_{m+n+1}) = Z \times \mathbb{A}^1$  in  $W \times \mathbb{A}^1$ . Thus the tuple  $h^d(c)$  is an element in  $\text{Fr}_n(U \times \mathbb{A}^1, X_+ \wedge T^{n+1})$  whose support equals  $Z \times \mathbb{A}^1$ .

**Remark 7.2.** Let  $c = (Z, W, \varphi; g) \in \text{Fr}_n(U, X_+ \wedge T^{n+1})$  and let  $\pi : W \rightarrow U$  be the composite map  $U \xleftarrow{pr_U} U \times \mathbb{A}^{m+n+1} \leftarrow W$ . Then the map  $(\varphi, \pi) : W \rightarrow \mathbb{A}^{m+n+1} \times U$  is quasi-finite over  $0 \times U$ . Hence it is quasi-finite over some Zariski open neighborhood  $V$  of  $0 \times U$ . Then  $W' = (\varphi, \pi)^{-1}(V)$  is a Zariski open neighborhood of  $Z$  in  $W$ . Replacing  $W$  by  $W'$  if necessary, we may and shall always assume in what follows that  $(\varphi, \pi) : W \rightarrow \mathbb{A}^{m+n+1} \times U$  is quasi-finite.

**Lemma 7.3.** *If  $U \in \text{AffSm}/k$  then the following statements are true:*

- (i) *if  $c \in \text{Fr}_m^{<d}(U, X_+ \wedge T^{n+1})$ , then  $t_d(c) \in \text{Fr}_m^{qf}(U, X_+ \wedge T^{n+1})$ ;*
- (ii) *if  $c \in (\text{Fr}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1})$ , then  $h_s^d(c) \in \text{Fr}_m^{qf}(U \times \mathbb{A}^1, X_+ \wedge T^{n+1})$ .*

<sup>1</sup>We thank A. Ananyevskiy for suggesting Lemma 7.1 in its present form.

*Proof.* Prove the first assertion. Let  $c = (Z, W, \varphi; g) \in \text{Fr}_m(U, X_+ \wedge T^{n+1}) \in \text{Fr}_m^{<d}(U, X_+ \wedge T^{n+1})$ . Let  $F_1, \dots, F_r \in k[\mathbb{A}_U^{m+n+1}]$  be a  $c$ -defining set with  $\deg F_i < d$  for all  $i = 1, \dots, r$ . We must check that  $t_d(c)$  is in  $\text{Fr}_m^{qf}(U, X_+ \wedge T^{n+1})$ . So, take

$$Y = Z(\varphi_1 - \varphi_{m+n+1}^d, \varphi_2 - \varphi_{m+n+1}^{d^2}, \dots, \varphi_{m+n} - \varphi_{m+n+1}^{d^{m+n}}) \subset W.$$

We must check that for any point  $u \in U$  the fiber  $Y(u)$  of  $Y$  over  $u$  is finite. Let  $\theta : \mathbb{A}^1 \rightarrow \mathbb{A}^{m+n+1}$  be a morphism taking a point  $t$  to the point  $(t^d, t^{d^2}, \dots, t^{d^{m+n}}, t)$ . It is a closed embedding with the image  $C = \theta(\mathbb{A}^1)$ .

By Remark 7.2 the morphism  $\psi = (\varphi, \pi) : W \rightarrow \mathbb{A}^{m+n+1} \times U$  is quasi-finite. Given a point  $u \in U$  there is a polynomial  $F$  from the  $c$ -defining set such that  $F(-, u)$  is non-zero and its vanishing locus  $Z(F(-, u))$  in  $\mathbb{A}_u^{m+n+1}$  contains  $\psi(W(u))$ . Clearly,  $Y(u)$  is contained in the set

$$\psi^{-1}(Z(F(-, u)) \cap C).$$

The set  $Z(F(-, u)) \cap C$  is in a bijection with the vanishing locus of the polynomial  $F(t^d, t^{d^2}, \dots, t^{d^{m+n}}, t)$  on the line  $\mathbb{A}^1$  with the coordinate  $t$ . Thus by Lemma 7.1 the set  $Z(F(-, u)) \cap C$  is finite, and hence so is  $Y(u)$ . The first assertion is proved.

Let us verify the second assertion. Let  $c = (Z, W, \varphi; g) \in (\text{Fr}_m^{qf})^{<d}(U, X \times T^{n+1})$  and let  $F_1, \dots, F_r \in k[\mathbb{A}_U^{m+n+1}]$  be a  $c$ -defining set with  $\deg F_i < d$  for all  $i = 1, \dots, r$ . We must check that  $h^d(c)$  is in  $\text{Fr}_m^{qf}(U \times \mathbb{A}^1, X \times T^{n+1})$ . So, take

$$Y_s = Z(\varphi_1 - s\varphi_{m+n+1}^d, \varphi_2 - s\varphi_{m+n+1}^{d^2}, \dots, \varphi_{m+n} - s\varphi_{m+n+1}^{d^{m+n}}) \subset W \times \mathbb{A}^1,$$

where  $s$  is the coordinate on the additional factor  $\mathbb{A}^1$ . We must check that for any point  $v \in U \times \mathbb{A}^1$  the fiber  $Y_s(v)$  of  $Y_s$  over  $v$  is finite. Replacing the base field  $k$  by its algebraic closure  $\bar{k}$ , we may assume that any point  $v \in U \times \mathbb{A}^1$  is of the form  $(u, a)$  with  $a \in \bar{k}$ . So, we must check that for any point  $(u, a) \in U \times \mathbb{A}^1$  the fiber  $Y_a(u)$  of  $Y_s$  over  $(u, a)$  is finite.

Given  $0 \neq a \in \bar{k}$ , let  $\theta_a : \mathbb{A}^1 \rightarrow \mathbb{A}^{m+n+1}$  be a morphism taking a point  $t$  to the point  $(at^d, at^{d^2}, \dots, at^{d^{m+n}}, t)$ . It is a closed embedding with the image  $C_a = \theta_a(\mathbb{A}^1)$ . For  $a = 0$  let  $\theta_0 : \mathbb{A}^1 \rightarrow \mathbb{A}^{m+n+1}$  be the morphism taking a point  $t$  to  $(0, 0, \dots, 0, t)$ . It is a closed embedding with the image  $C_0 = \theta_0(\mathbb{A}^1)$ . It is the last coordinate line  $\mathbb{A}_{m+n+1}^1$  in  $\mathbb{A}^{m+n+1}$ .

By Remark 7.2 the morphism  $\psi = (\varphi, \pi) : W \rightarrow \mathbb{A}^{m+n+1} \times U$  is quasi-finite. Given a point  $u \in U$ , there is a polynomial  $F$  from the  $c$ -defining set such that  $F(-, u)$  is non-zero and its vanishing locus  $Z(F(-, u))$  in  $\mathbb{A}_u^{m+n+1}$  contains  $\psi(W(u))$ . For a given  $0 \neq a \in \bar{k}$  the set  $Y_a(u)$  is contained in the set

$$\psi^{-1}(Z(F(-, u)) \cap C_a).$$

The set  $Z(F(-, u)) \cap C_a$  is in a bijection with the vanishing locus of the polynomial  $F(at^d, at^{d^2}, \dots, at^{d^{m+n}}, t)$  on the line  $\mathbb{A}^1$  with the coordinate  $t$ . Thus by Lemma 7.1 the set  $Z(F(-, u)) \cap C_a$  is finite in this case, and hence so is  $Y_a(u)$ .

For  $a = 0$ , the set  $Y_0(u)$  coincides with the closed subset  $Z(\varphi_1, \dots, \varphi_{m+n})$  in  $W$ . It is quasi-finite over  $U$ , because  $c = (Z, W, \varphi; g) \in (\text{Fr}_m^{qf})^{<d}(U, X_+ \wedge T^{n+1})$ . The second assertion is proved.  $\square$

Lemma 7.3 implies that the assignment  $c \mapsto t_d(c)$  gives a morphism of presheaves of pointed sets

$$t_d : \text{Fr}_m^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \text{Fr}_m^{qf}(-, X_+ \wedge T^{n+1})$$

on  $\text{AffSm}/k$ . It also implies that the assignment  $c \mapsto h^d(c)$  gives a morphism of presheaves of pointed sets

$$h_d : \text{Fr}_m^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \text{Fr}_m^{qf}(- \times \mathbb{A}^1, X_+ \wedge T^{n+1})$$



on  $AffSm/k$ . Finally, the assignment  $c \mapsto h_d(c)$  gives a morphism of presheaves of pointed sets

$$h_d^{qf} : (\text{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \text{Fr}_m^{qf}(- \times \mathbb{A}^1, X_+ \wedge T^{n+1})$$

on  $AffSm/k$ .

Consider a diagram

$$\begin{array}{ccc} \text{Fr}_m^{<d}(-, X_+ \wedge T^{n+1}) & \xrightarrow{i_d} & \text{Fr}_m(-, X_+ \wedge T^{n+1}) \\ \uparrow \text{in}_d & \searrow t_d & \uparrow \text{in} \\ (\text{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) & \xrightarrow{j_d} & \text{Fr}_m^{qf}(-, X_+ \wedge T^{n+1}) \end{array}$$

of presheaves of pointed sets on  $AffSm/k$ . Lemma 7.3 shows that  $h_d : \text{Fr}_m^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \text{Fr}_m(-, X_+ \wedge T^{n+1}))$  is an  $\mathbb{A}^1$ -homotopy between the morphisms  $i \circ t_d$  and  $i_d$ . It also shows that  $h_d^{qf} : (\text{Fr}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \text{Fr}_m^{qf}(-, X_+ \wedge T^{n+1}))$  is an  $\mathbb{A}^1$ -homotopy between the morphisms  $t_d \circ \text{in}_d$  and  $j_d$ .

Applying the free abelian group functor to the morphisms  $t_d, i_d, j_d, \text{in}_d$  and  $\text{in}$ , we get certain morphisms between presheaves of abelian groups as well as two  $\mathbb{A}^1$ -homotopies (namely,  $\mathbb{Z}[t_d]$ ,  $\mathbb{Z}[i_d]$ ,  $\mathbb{Z}[j_d]$ ,  $\mathbb{Z}[\text{in}_d]$ ,  $\mathbb{Z}[\text{in}]$ ,  $\mathbb{Z}[h_d]$  and  $\mathbb{Z}[h_d^{qf}]$ ). Note that these morphisms and these two homotopies respect the additivity relations. Thus following Definition 6.5, we finally get morphisms  $I_d, J_d, \text{In}_d, \text{In}$  and a morphism of presheaves

$$T_d : \mathbb{ZF}_m^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \mathbb{ZF}_m^{qf}(-, X_+ \wedge T^{n+1}),$$

and two  $\mathbb{A}^1$ -homotopies  $H_d, H_d^{qf}$ . In this way we get a diagram

$$\begin{array}{ccc} \mathbb{ZF}_m^{<d}(-, X_+ \wedge T^{n+1}) & \xrightarrow{I_d} & \mathbb{ZF}_m(-, X_+ \wedge T^{n+1}) \\ \uparrow \text{In}_d & \searrow T_d & \uparrow \text{In} \\ (\mathbb{ZF}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) & \xrightarrow{J_d} & \mathbb{ZF}_m^{qf}(-, X_+ \wedge T^{n+1}) \end{array}$$

of presheaves of abelian groups on  $AffSm/k$ .

We document these arguments as follows.

**Lemma 7.4.** *The  $\mathbb{A}^1$ -homotopy  $h_d$  yields an  $\mathbb{A}^1$ -homotopy*

$$H_d : \mathbb{ZF}_m^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \mathbb{ZF}_m(-, X_+ \wedge T^{n+1}))$$

*between  $I \circ T_d$  and  $I_d$ . The  $\mathbb{A}^1$ -homotopy  $h_d^{qf}$  yields an  $\mathbb{A}^1$ -homotopy*

$$H_d^{qf} : (\mathbb{ZF}_m^{qf})^{<d}(-, X_+ \wedge T^{n+1}) \rightarrow \underline{\text{Hom}}(\mathbb{A}^1, \mathbb{ZF}_m^{qf}(-, X_+ \wedge T^{n+1}))$$

*between  $T_d \circ \text{In}_d$  and  $J_d$ .*

**Proposition 7.5.** *For any integers  $m, n \geq 0$  and for any  $d > 0$  the morphism*

$$\text{In} : C_* \mathbb{ZF}_m^{qf}(-, X_+ \wedge T^{n+1}) \rightarrow C_* \mathbb{ZF}_m(-, X_+ \wedge T^{n+1})$$

*of complexes of presheaves of abelian groups is a section-wise quasi-isomorphism on the category  $AffSm/k$ .*

*Proof.* The functor  $C_*$  converts  $\mathbb{A}^1$ -homotopies to simplicial homotopies. Now the proposition follows from Lemma 7.4 and Corollary 6.6.  $\square$

We are now in a position to prove Theorem 1.5

*Proof of Theorem 1.5.* By Definitions 3.2 and 2.3 the complexes  $C_* \mathbb{Z}F^{qf}(-, X_+ \wedge T^{n+1})$  and  $C_* \mathbb{Z}F(-, X_+ \wedge T^{n+1})$  are the colimits of complexes  $C_* \mathbb{Z}F_m^{qf}(-, X_+ \wedge T^{n+1})$  and  $C_* \mathbb{Z}F_m(-, X_+ \wedge T^{n+1})$  over the suspension morphisms  $\Sigma$ . The morphisms  $In$  commute with the suspension morphisms  $\Sigma$ , i.e.  $\Sigma \circ In = In \circ \Sigma : C_* \mathbb{Z}F_m^{qf}(-, X_+ \wedge T^{n+1}) \rightarrow C_* \mathbb{Z}F_{m+1}(-, X_+ \wedge T^{n+1})$ . Proposition 7.5 now completes the proof.  $\square$

## 8. PROOF OF THEOREM 1.1

To finish the paper, it remains to prove Theorem 1.1. To this end we need some definitions and Theorem 1.2 stated in the Introduction. Also, we follow notation of Section 5.

Given  $X \in Sm/k$ ,  $n \geq 0$  and a finite pointed set  $(K, *)$ , put  $(X_+ \wedge T^n) \otimes K := (X \otimes K)_+ \wedge T^n$ . Set  $K' = K - *$ . Clearly,  $(X_+ \wedge T^n) \otimes K = (X \times K' \times \mathbb{A}^m) / (X \times K' \times \mathbb{A}^m - X \times K' \times \{0\})$ . For a  $k$ -smooth scheme  $U$  and a finite pointed set  $(K, *)$  set

$$\text{Fr}_n(U, (X_+ \wedge T^m) \otimes K) := \text{Fr}_n(U, (X \times K' \times \mathbb{A}^m) / (X \times K' \times \mathbb{A}^m - X \times K' \times \{0\})), \quad (8)$$

where the right hand side is from Definition 2.1(III). Also, set

$$\mathbb{Z}F_n(U, (X_+ \wedge T^m) \otimes K) := \mathbb{Z}F_n(U, (X \times K' \times \mathbb{A}^m) / (X \times K' \times \mathbb{A}^m - X \times K' \times \{0\})),$$

where the right hand side is from Definition 2.3. Finally set

$$F_n(U, (X_+ \wedge T^m) \otimes K) := F_n(U, (X \times K' \times \mathbb{A}^m) / (X \times K' \times \mathbb{A}^m - X \times K' \times \{0\})), \quad (9)$$

where the right hand side is from Definition 2.4. By Definition 2.4 the set  $F_n(U, (X_+ \wedge T^m) \otimes K) - 0_n$  is a free basis of the abelian group  $\mathbb{Z}F_n(U, (X_+ \wedge T^m) \otimes K)$ .

Denote by  $\mathbb{Z}\text{Fr}_*^{S^1}(X_+ \wedge T^n)$  the Segal  $S^1$ -spectrum

$$(\mathbb{Z}\text{Fr}_*(-, X_+ \wedge T^n), \mathbb{Z}\text{Fr}_*(-, (X_+ \wedge T^n) \otimes S^1), \dots).$$

Also, let  $EM(\mathbb{Z}F_*(-, X_+ \wedge T^n))$  be the Segal  $S^1$ -spectrum

$$(\mathbb{Z}F_*(-, X_+ \wedge T^n), \mathbb{Z}F_*(-, (X_+ \wedge T^n) \otimes S^1), \dots).$$

The equality  $\mathbb{Z}F_*(-, (X \sqcup X')_+ \wedge T^n) = \mathbb{Z}F_*(-, X_+ \wedge T^n) \oplus \mathbb{Z}F_*(-, X'_+ \wedge T^n)$  shows that the  $\Gamma$ -space  $(K, *) \mapsto \mathbb{Z}F_*(U, (X_+ \wedge T^n) \otimes K)$  is fully determined by the abelian group  $\mathbb{Z}F_*(U, X_+ \wedge T^n)$ . Hence  $EM(\mathbb{Z}F_*(-, X_+ \wedge T^n))$  is the Eilenberg–Mac Lane spectrum for  $\mathbb{Z}F_*(-, X_+ \wedge T^n)$ . The  $\Gamma$ -space morphism  $[(K, *) \mapsto \mathbb{Z}\text{Fr}_*(-, (X_+ \wedge T^n) \otimes K)] \rightarrow [(K, *) \mapsto \mathbb{Z}F_*(-, (X_+ \wedge T^n) \otimes K)]$  induces a morphism of framed  $S^1$ -spectra

$$\lambda_{X_+ \wedge T^n} : \mathbb{Z}\text{Fr}_*^{S^1}(X_+ \wedge T^n) \rightarrow EM(\mathbb{Z}F_*(-, X_+ \wedge T^n)).$$

Also, denote by  $\mathbb{Z}M_{fr}(X \times T^n)$ ,  $X \in Sm/k$ , the Segal  $S^1$ -spectrum

$$(C_* \mathbb{Z}\text{Fr}(-, X_+ \wedge T^n), C_* \mathbb{Z}\text{Fr}(-, (X_+ \wedge T^n) \otimes S^1), \dots). \quad (10)$$

Let  $LM_{fr}(X \times T^n)$  be the Segal  $S^1$ -spectrum

$$EM(\mathbb{Z}F(\Delta^\bullet \times -, X_+ \wedge T^n)) = (\mathbb{Z}F(\Delta^\bullet \times -, X_+ \wedge T^n), \mathbb{Z}F(\Delta^\bullet \times -, (X_+ \wedge T^n) \otimes S^1), \dots).$$

The above arguments show that  $LM_{fr}(X \times T^n)$  is the Eilenberg–Mac Lane spectrum associated with the complex  $\mathbb{Z}F(\Delta^\bullet \times -, X_+ \wedge T^n)$ . The  $\Gamma$ -space morphism

$$[(K, *) \mapsto \mathbb{Z}\text{Fr}(\Delta^\bullet \times -, (X_+ \wedge T^n) \otimes K)] \rightarrow [(K, *) \mapsto \mathbb{Z}F(\Delta^\bullet \times -, (X_+ \wedge T^n) \otimes K)]$$

induces a morphism of framed  $S^1$ -spectra

$$l_{X \times T^n} : \mathbb{Z}M_{fr}(X \times T^n) \rightarrow LM_{fr}(X \times T^n).$$

Note that stable homotopy groups of  $LM_{fr}(X \times T^n) = EM(\mathbb{Z}F(\Delta^\bullet \times -, X_+ \wedge T^n))$  are equal to homology groups of the complex  $\mathbb{Z}F(\Delta^\bullet \times -, X_+ \wedge T^n)$ . By [Sch, §II.6.2] homotopy groups  $\pi_*(\mathbb{Z}M_{fr}(X \times T^n)(U))$  of  $\mathbb{Z}M_{fr}(X \times T^n)$  evaluated at  $U \in Sm/k$  are homology groups  $H_*(M_{fr}(X \times T^n)(U))$  of  $M_{fr}(X \times T^n)(U)$ .

For the convenience of the reader we recall Theorem 1.2 proved in Appendix B. It computes, in particular, homology of the framed motives  $M_{fr}(X \times T^n)$  of the relative motivic spheres  $X_+ \wedge T^n$ .

**Theorem 1.2.** *For any integer  $m \geq 0$ , the natural morphism of framed  $S^1$ -spectra*

$$\lambda_{X_+ \wedge T^m} : \mathbb{Z}Fr_*^{S^1}(X_+ \wedge T^m) \rightarrow EM(\mathbb{Z}F_*(-, X_+ \wedge T^m))$$

*is a schemewise stable equivalence. Moreover, the natural morphism of framed  $S^1$ -spectra*

$$l_{X \times T^m} : \mathbb{Z}M_{fr}(X \times T^m) \rightarrow LM_{fr}(X \times T^m)$$

*is a schemewise stable equivalence. In particular, for any  $U \in Sm/k$  one has*

$$\pi_*(\mathbb{Z}M_{fr}(X \times T^m)(U)) = H_*(\mathbb{Z}F(\Delta^\bullet \times U, X_+ \wedge T^m)) = H_*(C_*\mathbb{Z}F(U, X_+ \wedge T^m)).$$

By [GP1, Def. 9.1] the framed motive of the pointed sheaf  $X_+ \wedge T^n$  is the Segal  $S^1$ -spectrum

$$M_{fr}(X \times T^n) = (C_*Fr_\bullet(-, X_+ \wedge T^n), C_*Fr_\bullet(-, (X_+ \wedge T^n) \otimes S^1), C_*Fr_\bullet(-, (X_+ \wedge T^n) \otimes S^2), \dots).$$

As explained in the proof of [GP1, 9.2] there is a natural bijection

$$Fr_m(U, X_+ \wedge T^n) \rightarrow Fr_{m,\bullet}(U, X_+ \wedge T^n) := \text{Hom}_{Sh_{\bullet}^{nis}(Sm/k)}(U_+ \wedge (\mathbb{P}^1, \infty)^{\wedge m}, X_+ \wedge T^n \wedge T^m).$$

Thus  $M_{fr}(X \times T^n)$  is the Segal  $S^1$ -spectrum of the form

$$(C_*Fr(-, X_+ \wedge T^n), C_*Fr(-, (X_+ \wedge T^n) \otimes S^1), C_*Fr(-, (X_+ \wedge T^n) \otimes S^2), \dots)$$

with the presheaves  $Fr(-, X_+ \wedge T^n)$  defined in Section 5. In particular,  $\mathbb{Z}M_{fr}(X \times T^n)$  is exactly the Segal  $S^1$ -spectrum (10) defined above in the present section.

We are now in a position to prove the remaining Theorem 1.1.

*Proof of Theorem 1.1.* By Theorem 1.3 the map of complexes of presheaves of abelian groups

$$C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1)/C_*\mathbb{Z}F(X_+ \wedge T^n \times \mathbb{G}_{m+}) \rightarrow C_*\mathbb{Z}F(X_+ \wedge T^{n+1}) \quad (11)$$

is a local quasi-isomorphism. The  $S^1$ -spectra  $LM_{fr}(X \times T^n \times \mathbb{A}_+^1)$ ,  $LM_{fr}(X \times T^n \times \mathbb{G}_m)$  and  $LM_{fr}(X \times T^{n+1})$  are the Eilenberg-MacLane  $S^1$ -spectra of the complexes  $C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{A}_+^1)$ ,  $C_*\mathbb{Z}F(X_+ \wedge T^n \wedge \mathbb{G}_{m+})$  and  $C_*\mathbb{Z}F(X_+ \wedge T^{n+1})$  respectively. Thus the map

$$LM_{fr}(X \times T^n \times \mathbb{A}_+^1)/LM_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow LM_{fr}(X \times T^{n+1}),$$

induced by (11), is a local stable weak equivalence, and hence so is the map

$$\mathbb{Z}M_{fr}(X \times T^n \times \mathbb{A}_+^1)/\mathbb{Z}M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow \mathbb{Z}M_{fr}(X \times T^{n+1})$$

by Theorem 1.2. The  $S^1$ -spectra  $M_{fr}(X \times T^n \times \mathbb{A}_+^1)$ ,  $M_{fr}(X \times T^n \times \mathbb{G}_m)$ ,  $M_{fr}(X \times T^{n+1})$  are connected. Now the stable Whitehead theorem [Sch, II.6.30] implies the map

$$M_{fr}(X \times T^n \times \mathbb{A}_+^1)/M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow M_{fr}(X \times T^{n+1}) \quad (12)$$

is a local stable weak equivalence. Consider the following sequence of natural maps

$$\begin{aligned}
M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge(n+1)}) &= M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n} \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)) \xrightarrow{(1)} \\
&\text{Cone}[M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n} \times \mathbb{G}_m) \rightarrow M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n} \times \mathbb{A}^1)] \xrightarrow{(2)} \\
&\text{Cone}[M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow M_{fr}(X \times T^n \times \mathbb{A}^1)] \xrightarrow{(3)} \\
&M_{fr}(X \times T^n \times \mathbb{A}^1) / M_{fr}(X \times T^n \times \mathbb{G}_m) \xrightarrow{(4)} M_{fr}(X \times T^{n+1}).
\end{aligned}$$

The arrows (1) and (3) are sectionwise stable weak equivalences by standard reasons. The arrow (2) is a local stable weak equivalence by induction. The arrow (4) is exactly the map (12), and so it is a local stable weak equivalence.

Hence for any  $\ell \geq 1$  the canonical morphism

$$M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge \ell}) \rightarrow M_{fr}(X \times T^\ell) \quad (13)$$

is a local stable weak equivalence. By [GP1, 9.2] the simplicial  $S^1$ -spectrum  $M_{fr}(X \times T^\ell)(U)$  with  $U$  a local Henselian smooth scheme and  $X$  any smooth scheme is a connected  $\Omega$ -spectrum. The proof of [GP1, 9.2] works equally for the  $\Gamma$ -space  $K \rightarrow C_* \text{Fr}(U, (X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge \ell}) \otimes K)$ . Thus the latter  $\Gamma$ -space is special. By Lemma A.1 the zero space  $C_* \text{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge \ell})$  of the Segal  $S^1$ -spectrum  $M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge \ell})(U)$  is connected. Thus the Segal  $S^1$ -spectrum  $M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge \ell})(U)$  is a connected  $\Omega$ -spectrum. Hence the morphism (13) is a local level weak equivalence.

Finally, using Theorem 1.2, the assertion that the sequence of  $S^1$ -spectra

$$M_{fr}(X \times T^n \times \mathbb{G}_m) \rightarrow M_{fr}(X \times T^n \times \mathbb{A}^1) \rightarrow M_{fr}(X \times T^{n+1})$$

is locally a homotopy cofiber sequence in the Nisnevich topology reduces to the assertion that the sequence of complexes of linear framed presheaves

$$C_* \mathbb{Z}\text{F}(X_+ \wedge T^n \wedge \mathbb{G}_{m+}) \rightarrow C_* \mathbb{Z}\text{F}(X_+ \wedge T^n \wedge \mathbb{A}_+^1) \rightarrow C_* \mathbb{Z}\text{F}(X_+ \wedge T^{n+1})$$

is locally a homotopy cofiber sequence. The latter follows from Theorem 1.3.  $\square$

**Corollary 8.1.** *For every  $n \geq 0$ , the natural morphism*

$$M_{fr}(X \times T^n \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)) \rightarrow M_{fr}(X \times T^{n+1})$$

*is locally a level weak equivalence of  $S^1$ -spectra in the Nisnevich topology.*

*Proof.* Consider a commutative diagram

$$\begin{array}{ccc}
M_{fr}(X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge(n+1)}) & \longrightarrow & M_{fr}(X \times T^{n+1}) \\
\downarrow & & \parallel \\
M_{fr}(X \times T^n \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)) & \longrightarrow & M_{fr}(X \times T^{n+1})
\end{array}$$

The left vertical and upper horizontal arrows are locally level weak equivalences of  $S^1$ -spectra by Theorem 1.1, and hence so is the lower horizontal arrow.  $\square$

## APPENDIX A.

In this section we prove the following useful

**Lemma A.1.** *For any  $X \in Sm/k$  and any  $n > 0$  the simplicial pointed presheaves  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n})$  and  $C_*\mathrm{Fr}(-, X_+ \wedge T^n)$  are locally connected in the Nisnevich topology.*

*Proof.* Firstly check the Nisnevich local connectivity of  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m))$ . Clearly, the map  $\pi_0(C_*\mathrm{Fr}(-, X \times \mathbb{A}^1)) \rightarrow \pi_0(C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)))$  is surjective. On the other hand the composite map of pointed sets

$$\pi_0(C_*\mathrm{Fr}(-, X \times \mathbb{G}_m)) \rightarrow \pi_0(C_*\mathrm{Fr}(-, X \times \mathbb{A}^1)) \rightarrow \pi_0(C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)))$$

is constant, because it factors through the pointed set  $\pi_0(C_*\mathrm{Fr}(-, (X \times \mathbb{G}_m) \otimes I)) = *$ . Thus it is sufficient to check that for any local essentially  $k$ -smooth Henselian  $U$  the map

$$\pi_0(C_*\mathrm{Fr}(U, X \times \mathbb{G}_m)) \rightarrow \pi_0(C_*\mathrm{Fr}(U, X \times \mathbb{A}^1)) \quad (14)$$

is surjective. Take a framed correspondence

$$c_0 = (Z, W, \varphi; (f, g) : W \rightarrow X \times \mathbb{A}^1) \in \mathrm{Fr}_n(U, X \times \mathbb{A}^1).$$

We may and will assume that  $W = (\mathbb{A}_U^n)_Z^h$  is the henselization of  $\mathbb{A}_U^n$  at the closed subset  $Z$ . We want to find  $h_t \in \mathrm{Fr}_n(\mathbb{A}_U^1, \mathbb{A}_k^1)$  and  $c_1 \in \mathrm{Fr}_n(U, X \times \mathbb{G}_{m,k})$  such that  $h_0 = c_0$ ,  $h_1 = j \circ c_1$ , where  $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}_k^1$  is the open embedding. Firstly, suppose the residue field of  $U$  at its closed point  $u$  is finite over the ground field  $k$ . Let  $Z_u$  be the fiber of  $Z$  over  $u$ . Since  $Z$  is finite over  $U$  the scheme  $Z_u$  is finite over  $\mathrm{pt} := \mathrm{Spec}(k)$ .

Hence one can take a  $k$ -rational point  $a$  from the open subset  $\mathbb{A}_k^1 - g(Z_u) - \{0\}$  of the affine line  $\mathbb{A}_k^1$ . Then  $(g - a)(Z_u) \subset \mathbb{G}_{m,k}$  and  $(g - a)(W) \subset \mathbb{G}_{m,k}$ , hence  $(f, (g - a))(W) \subset X \times \mathbb{G}_{m,k}$  as well. Thus  $c_a := (Z, W, \varphi; (f, (g - a))) \in \mathrm{Fr}_n(\mathrm{pt}, X \times \mathbb{G}_{m,k})$ . Set

$$h_t = (Z \times \mathbb{A}^1, W \times \mathbb{A}^1, \varphi; (f, (g - t))) \in \mathrm{Fr}_n(\mathbb{A}^1, X \times \mathbb{A}_k^1).$$

Clearly,  $h_0 = c_0$ ,  $h_a = j \circ c_a$ . Hence the arrow (14) is surjective provided that the field  $k(u)$  is finite over the ground field  $k$ .

Note that for any field extension  $K/k$ , any  $k$ -smooth scheme  $X$  and any  $K$ -smooth scheme  $V$  one has an adjunction  $\mathrm{Fr}(V, X) = \mathrm{Fr}(V, X_K)$ . Using this adjunction, we may always assume that the residue field  $k(u)$  is finite over the ground field. Thus the arrow (14) is surjective and  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m))$  is locally connected.

By induction, suppose  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n})$  is locally connected. Then  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge(n+1)})$  is the realization of a simplicial space of the form

$$[r] \mapsto C_*\mathrm{Fr}(-, Y_r \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n})$$

with  $Y_r \in Sm/k$ . Since each  $C_*\mathrm{Fr}(-, Y_r \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge n})$  is locally connected by the induction hypothesis, then  $C_*\mathrm{Fr}(-, X \times (\mathbb{A}^1 \rfloor \mathbb{G}_m)^{\wedge(n+1)})$  is locally connected as well.

Now let us prove the Nisnevich local connectivity of  $C_*\mathrm{Fr}(-, X_+ \wedge T^n)$ . We give a proof for  $n = 1$ . The general case is treated similarly. For any local essentially  $k$ -smooth Henselian  $U$  consider a framed correspondence

$$b_0 = (Z, W, \varphi_1, \dots, \varphi_n, \varphi_{n+1}; f : W \rightarrow X) \in \mathrm{Fr}_n(U, X_+ \wedge T).$$

It is sufficient to find  $h_t \in \mathrm{Fr}_n(\mathbb{A}_U^1, X_+ \wedge T)$  such that  $h_0 = b_0$  and  $h_1 = 0_n$  is the empty framed correspondence. We may and will assume that  $W = (\mathbb{A}_U^n)_Z^h$  is the henselization of  $\mathbb{A}_U^n$  at the closed subset  $Z$ . As above first suppose the residue field of  $U$  at its closed point  $u$  is finite over

the ground field  $k$ . Let  $Z_u$  be the fiber of  $Z$  over  $u$ . Take a  $k$ -rational point  $a$  from the open subset  $\mathbb{A}_k^1 - \varphi_{n+1}(Z_u) - \{0\}$  of the affine line  $\mathbb{A}_k^1$ . Then  $(\varphi_{n+1} - a)(Z_u) \subset \mathbb{G}_{m,k}$ . Set

$$h_t = (Z \times \mathbb{A}^1, W \times \mathbb{A}^1, \varphi_1, \dots, \varphi_n, \varphi_{n+1} - t; f : W \rightarrow X) \in \text{Fr}_n(\mathbb{A}_U^1, X_+ \wedge T).$$

Clearly,  $h_0 = b_0$ ,  $h_a = 0_n$ .

Using the above adjunction  $\text{Fr}(V, Y) = \text{Fr}(V, Y_K)$ , we may always assume that the residue field  $k(u)$  is finite over the ground field. Thus  $C_* \text{Fr}(-, X_+ \wedge T)$  is locally connected. The same is true for  $C_* \text{Fr}(-, X_+ \wedge T^n)$ , hence the lemma.  $\square$

## APPENDIX B.

The main goal of this section is to prove Theorem 1.2. It will be proved at the end of the section.

Let  $\mathbb{S}$  be the sphere  $S^1$ -spectrum. Let  $*$   $\subset \mathbb{S}$  be its trivial  $S^1$ -subspectrum corresponding to the basepoint. Let  $\mathcal{A}$  be a pointed set with a distinguished point  $*$ . Denote by  $\mathbb{S}_{\mathcal{A}}$  the  $S^1$ -spectrum  $\prod_{(\mathcal{A}-*)} \mathbb{S}$ . Let  $\mathbb{S}'_{\mathcal{A}}$  be the  $S^1$ -subspectrum  $\vee_{(\mathcal{A}-*)} \mathbb{S}$  in  $\mathbb{S}_{\mathcal{A}}$ .

Given a finite pointed subset  $A \subset \mathcal{A}$ , let  $\mathbb{S}_A \subset \mathbb{S}_{\mathcal{A}}$  be an  $S^1$ -subspectrum of the form  $\prod_{(A-*)} E_a$ , where  $E_a = \mathbb{S}$  if  $a \in A - *$  and  $E_a = *$  if  $a \in \mathcal{A} - A$ . If  $a \in A - *$  we shall write  $\mathbb{S}_a$  to denote  $\mathbb{S}_{\{a, *\}}$ , where  $\{a, *\} \subset A$  is the two elements subset of  $A$ . Let  $\mathbb{S}'_A \subset \mathbb{S}_A$  be the  $S^1$ -subspectrum  $\vee_{a \in (A-*)} \mathbb{S}_a$  in  $\mathbb{S}_A$ . Clearly, the inclusion  $\mathbb{S}'_A \subset \mathbb{S}_A$  is a stable equivalence of  $S^1$ -spectra. Set  $\mathbb{S}_{\mathcal{A}}^f := \cup_{A \subset \mathcal{A}} \mathbb{S}_A \subset \mathbb{S}_{\mathcal{A}}$ , where the union is taken over the set of all finite pointed subsets  $A$  of the pointed set  $\mathcal{A}$ .

The following lemma is straightforward and the proof is left to the reader.

**Lemma B.1.** *Let  $\mathcal{A}$  be a pointed set and  $I$  be a set such that for any  $i \in I$  there is a finite pointed subset  $A(i) \subset \mathcal{A}$ . We have two  $S^1$ -subspectra  $\cup_{i \in I} \mathbb{S}'_{A(i)}$ ,  $\cup_{i \in I} \mathbb{S}_{A(i)}$  of the spectrum  $\mathbb{S}_{\mathcal{A}}^f$ . Suppose  $\cup_{i \in I} A(i) = \mathcal{A}$ . Then  $\vee_{(\mathcal{A}-*)} \mathbb{S}_a = \cup_{i \in I} \mathbb{S}'_{A(i)}$  and the inclusion*

$$\vee_{(\mathcal{A}-*)} \mathbb{S}_a = \cup_{i \in I} \mathbb{S}'_{A(i)} \hookrightarrow \cup_{i \in I} \mathbb{S}_{A(i)}$$

*is a stable equivalence of  $S^1$ -spectra.*

An application of this lemma is given below in this section. For a finite pointed set  $(K, *)$  consider a set  $\text{Map}_{\bullet}^f(\mathcal{A}, K)$  of those maps  $\rho$  of pointed sets such that the set  $\rho^{-1}(K - *)$  is finite. Consider a  $\Gamma$ -space  $\Gamma_{\mathcal{A}}^f$  defined by  $\Gamma_{\mathcal{A}}^f(K, *) = \text{Map}_{\bullet}^f(\mathcal{A}, K)$ . For a finite pointed subset  $A \subset \mathcal{A}$  consider a subset  $\text{Map}_{\bullet}^A(\mathcal{A}, K) \subset \text{Map}_{\bullet}^f(\mathcal{A}, K)$  consisting of all maps  $\rho$  such that  $\rho^{-1}(K - *) \subset A$ . Consider a  $\Gamma$ -space  $\Gamma_A$  defined by  $\Gamma_A(K, *) = \text{Map}_{\bullet}^A(\mathcal{A}, K)$ .

Clearly, for any inclusion of finite pointed subsets  $A' \subset A$  of  $\mathcal{A}$  one has inclusions  $\text{Map}_{\bullet}^{A'}(\mathcal{A}, K) \subset \text{Map}_{\bullet}^A(\mathcal{A}, K)$  and  $\Gamma_{A'} \subset \Gamma_A$ . Moreover, one has

$$\cup_{A \subset \mathcal{A}} \text{Map}_{\bullet}^A(\mathcal{A}, K) = \text{Map}_{\bullet}^f(\mathcal{A}, K) \text{ and } \cup_{A \subset \mathcal{A}} \Gamma_A = \Gamma_{\mathcal{A}}^f,$$

where the union is taken over the set of all pointed finite subsets  $A$  in the pointed set  $\mathcal{A}$ .

Let  $A \subset \mathcal{A}$  be a finite pointed subset. For any element  $a \in A - *$  set  $\text{Map}_{\bullet}^a(\mathcal{A}, K) = \text{Map}_{\bullet}^{a \sqcup *}(\mathcal{A}, K)$ , where  $a \sqcup *$  stands for the two elements pointed subset of  $\mathcal{A}$ . Set  $\Gamma_a = \Gamma_{a \sqcup *}$ . That is  $\Gamma_a(K, *) = \text{Map}_{\bullet}^{a \sqcup *}(\mathcal{A}, K)$ . Let  $\text{Map}_{\bullet}^{A, s}(\mathcal{A}, K) \subset \text{Map}_{\bullet}^A(\mathcal{A}, K)$  consist of maps  $\rho$  such that the subset  $\rho^{-1}(K - *) \subset A$  either has one element or is the empty set. Consider a  $\Gamma$ -subspace  $\Gamma'_A \subset \Gamma_A$  such that  $\Gamma'_A(K, *) = \text{Map}_{\bullet}^{A, s}(\mathcal{A}, K)$ .

The  $\Gamma$ -space  $\Gamma_A$  is isomorphic to the  $\Gamma$ -space  $\prod_{a \in (A-*)} \Gamma_a$ . The  $\Gamma$ -space  $\Gamma'_A$  is isomorphic to the  $\Gamma$ -space  $\vee_{a \in (A-*)} \Gamma'_a$ . Moreover these isomorphisms are consistent with the inclusion  $\vee_{a \in (A-*)} \Gamma_a \subset \prod_{a \in (A-*)} \Gamma_a$ .



**Lemma B.2.** Let  $\mathcal{A}$ ,  $I$  and  $A(i)$  be as in Lemma B.1. There are two  $\Gamma$ -subspaces  $\cup_{i \in I} \Gamma'_{A(i)}$ ,  $\cup_{i \in I} \Gamma_{A(i)}$  of the  $\Gamma$ -space  $\Gamma_{\mathcal{A}}^f$ . Suppose  $\cup_{i \in I} A(i) = \mathcal{A}$ . Then  $\vee_{a \in (\mathcal{A} - *)} \Gamma_a = \cup_{i \in I} \Gamma'_{A(i)}$  and we have natural inclusions

$$\vee_{a \in (\mathcal{A} - *)} \Gamma_a = \cup_{i \in I} \Gamma'_{A(i)} \hookrightarrow \cup_{i \in I} \Gamma_{A(i)}$$

of  $\Gamma$ -spaces.

Let  $\text{Seg} : \Gamma\text{-spaces} \rightarrow S^1\text{-spectra}$  be the functor associating the Segal  $S^1$ -spectrum to a  $\Gamma$ -space. Then  $\text{Seg}(\Gamma_{\mathcal{A}}^f) = \mathbb{S}_{\mathcal{A}}^f$ . Given a non-distinguished element  $a \in \mathcal{A}$  one has  $\text{Seg}(\Gamma_a) = \mathbb{S}$  and the functor  $\text{Seg}$  converts the inclusion  $\Gamma_a \subset \Gamma_{\mathcal{A}}^f$  to the inclusion  $\mathbb{S}_a \subset \mathbb{S}_{\mathcal{A}}^f$ . For any finite pointed subset  $A$  in  $\mathcal{A}$ , the functor  $\text{Seg}$  converts the inclusion  $\Gamma_A \subset \Gamma_{\mathcal{A}}^f$  to the inclusion  $\mathbb{S}_A \subset \mathbb{S}_{\mathcal{A}}^f$ . It also converts the inclusion  $\Gamma'_A \subset \Gamma_A$  to the inclusion  $\mathbb{S}'_A \subset \mathbb{S}_A$  as well as the inclusion  $\vee_{a \in (\mathcal{A} - *)} \Gamma_a \subset \Gamma_{\mathcal{A}}^f$  to the inclusion  $\vee_{a \in (\mathcal{A} - *)} \mathbb{S}_a \subset \mathbb{S}_{\mathcal{A}}^f$ .

The above arguments together with Lemmas B.1 and B.2 prove the following

**Lemma B.3.** Let  $\mathcal{A}$ ,  $I$  and  $A(i)$  be as in Lemma B.1. There are two  $S^1$ -subspectra  $\cup_{i \in I} \text{Seg}(\Gamma'_{A(i)})$ ,  $\cup_{i \in I} \text{Seg}(\Gamma_{A(i)})$  of the  $S^1$ -spectrum  $\text{Seg}(\Gamma_{\mathcal{A}}^f)$ . Suppose  $\cup_{i \in I} A(i) = \mathcal{A}$ . Then  $\vee_{a \in (\mathcal{A} - *)} \text{Seg}(\Gamma_a) = \cup_{i \in I} \text{Seg}(\Gamma'_{A(i)})$  and the inclusion

$$\vee_{a \in (\mathcal{A} - *)} \text{Seg}(\Gamma_a) = \cup_{i \in I} \text{Seg}(\Gamma'_{A(i)}) \hookrightarrow \cup_{i \in I} \text{Seg}(\Gamma_{A(i)})$$

is a stable equivalence of  $S^1$ -spectra.

**Notation B.4.** Let  $U, X \in \text{Sm}/k$  and let  $m, n \geq 0$  be integers. Set  $\mathcal{A} = \text{Fr}_m(U, X_+ \wedge T^n)$  and regard it as a pointed set pointed by the empty framed correspondence  $0_m$ . Set  $I = \text{Fr}_m(U, X_+ \wedge T^n) - 0_m$ .

In the remaining part of this section we use notation from Section 5.

**Definition B.5.** Given  $\Phi = (Z, W, \varphi; g), \Phi' = (Z', W', \varphi'; g') \in \text{Fr}_m(U, X_+ \wedge T^n)$ , we write  $\Phi' \leq \Phi$  if there is a closed subset  $Z''$  in  $\mathbb{A}^m \times U$  such that  $Z = Z' \sqcup Z''$  and

$$(Z', W', \varphi'; g') = (Z', W - Z'', \varphi|_{W - Z''}; g|_{W - Z''}) \in \text{Fr}_m(U, X_+ \wedge T^n).$$

For any  $\Phi \in \text{Fr}_m(U, X_+ \wedge T^n)$  set  $A(\Phi) = \{\Psi \in \text{Fr}_m(U, X_+ \wedge T^n) : \Psi \leq \Phi\} \subset \mathcal{A}$ . Clearly,

$$\mathcal{A} = \cup_{\Phi \in I} A(\Phi).$$

For a finite pointed set  $(K, *)$  the pointed set  $\text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$  is defined by the formula (8). Let  $K' = K - *$ . By Definition 2.1(III) the set  $\text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$  consists of equivalence classes of tuples  $(Z, W, \varphi; g; f)$ , where  $Z$  is a closed subset of  $U \times \mathbb{A}^m$ , finite over  $U$ ,  $W$  is an étale neighborhood of  $Z$  in  $U \times \mathbb{A}^m$ ,  $\varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n}$  are regular functions on  $W$ ,  $(g, f) : W \rightarrow X \times K'$  is a regular map such that  $Z = Z(\varphi_1, \dots, \varphi_{m+n})$ . Notice that regular maps from  $W$  to  $X \otimes K$  are in one-to-one correspondence with couples of regular maps  $(W \rightarrow X, W \rightarrow K')$ .

For a finite pointed set  $(K, *)$ , the pointed set  $\text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$  is defined by the formula (9). By Definition 2.4 it consists of those elements  $(Z, W, \varphi; g; f) \in \text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$  such that the closed subset  $Z$  of  $U \times \mathbb{A}^m$  is connected.

**Definition B.6.** Denote by  $\Gamma_m(U, X_+ \wedge T^n)$  the  $\Gamma$ -space  $(K, *) \mapsto \text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$ . Similarly,  $\Gamma'_m(U, X_+ \wedge T^n)$  stands for the  $\Gamma$ -space  $(K, *) \mapsto \text{Fr}_m(U, (X_+ \wedge T^n) \otimes K)$ .

Given  $\Phi \in I$  define  $\Gamma_m(U, X_+ \wedge T^n)_{\Phi}$  as a  $\Gamma$ -subspace of the  $\Gamma$ -space  $\Gamma_m(U, X_+ \wedge T^n)$  such that for a finite pointed set  $(K, *)$

$$\Gamma_m(U, X_+ \wedge T^n)_{\Phi}(K) = \{(Z, W, \varphi; g; f) \in \text{Fr}_m(U, (X_+ \wedge T^n) \otimes K) \mid (Z, W, \varphi; g) \leq \Phi \in \text{Fr}_m(U, X_+ \wedge T^n)\}.$$

Define  $\Gamma'_m(U, X_+ \wedge T^n)_\Phi$  as a  $\Gamma$ -subspace of the  $\Gamma$ -space  $\Gamma'_m(U, X_+ \wedge T^n)$  such that for a finite pointed set  $(K, *)$

$$\Gamma'_m(U, X_+ \wedge T^n)_\Phi(K) = \{(Z, W, \varphi; g; f) \in F_n(U, (X_+ \wedge T^n) \otimes K) \mid (Z, W, \varphi; g) \leq \Phi \in F_m(U, X_+ \wedge T^n)\}.$$

**Definition B.7.** For a finite pointed set  $(K, *)$  put  $K' = K - *$  and consider a pointed set map

$$inc_K : Fr_m(U, (X_+ \wedge T^n) \otimes K) \rightarrow Map_{Sets_*}^f(\mathcal{A}, K),$$

which is defined as follows. Let  $\Psi = (Z, W, \varphi; g; f) \in Fr_m(U, (X_+ \wedge T^n) \otimes K)$  and  $a = (Z_a, W_a, \varphi_a; g_a) \in \mathcal{A} = F_m(U, X_+ \wedge T^n)$ . If the element  $a$  is in  $\mathcal{A} - A((Z, W, \varphi; g))$ , then the map  $inc_K(\Psi)$  takes the element  $a$  to the distinguished point  $*$  of the set  $K$ . If  $a \in A((Z, W, \varphi; g)) - 0_m$ , then the map  $inc_K(\Psi)$  takes the element  $a$  to  $f(Z_a) \in K' \subset K$ . Finally, the map  $inc_K(\Psi)$  sends  $0_m$  to the distinguished point  $*$  of the set  $K$ .

Recall that  $Z_a$  is connected and if  $a \in A((Z, W, \varphi; g))$ , then  $Z = Z_a \sqcup Z''$  for some  $Z''$ . Define a  $\Gamma$ -space morphism

$$inc_m : \Gamma_m(U, X_+ \wedge T^n) \rightarrow \Gamma_{\mathcal{A}}^f$$

sending a finite pointed set  $(K, *)$  to the pointed set map  $inc_K$ . It is straightforward to check that it is indeed a  $\Gamma$ -space morphism.

The following lemma is crucial.

**Lemma B.8.** *The  $\Gamma$ -space morphism  $inc_m$  is injective. Moreover, using this inclusion the following identifications hold:*

- (1) *for any  $\Phi \in I$ , one has  $\Gamma_m(U, X_+ \wedge T^n)_\Phi = \Gamma_{A(\Phi)}$  and  $\cup_{\Phi \in I} \Gamma_m(U, X_+ \wedge T^n)_\Phi = \cup_{\Phi \in I} \Gamma_{A(\Phi)}$ ;*
- (2) *for any  $\Phi \in I$ , one has  $\Gamma'_m(U, X_+ \wedge T^n)_\Phi = \Gamma'_{A(\Phi)}$  and  $\cup_{\Phi \in I} \Gamma'_m(U, X_+ \wedge T^n)_\Phi = \cup_{\Phi \in I} \Gamma'_{A(\Phi)}$ ;*
- (3) *for any  $a \in \mathcal{A} - 0_m = F_m(U, X_+ \wedge T^n) - 0_m$ , one has  $\Gamma_m(U, X_+ \wedge T^n)_a = \Gamma_a$ ;*
- (4)  $\cup_{a \in (\mathcal{A} - *)} \Gamma_m(U, X_+ \wedge T^n)_a = \cup_{a \in (\mathcal{A} - *)} \Gamma_a$ .

Applying the Segal functor  $Seg$ , we see that Lemmas B.2 and B.3 imply the following

**Proposition B.9.** *Let  $\mathcal{A} = F_m(U, X_+ \wedge T^n)$ ,  $I = Fr_m(U, X_+ \wedge T^n) - 0_m$  be as Notation B.4 and for  $\Phi \in I$  let the subset  $A(\Phi) \subset \mathcal{A}$  be as in Definition B.5. There are two  $S^1$ -subspectra  $\cup_{i \in I} Seg(\Gamma'_m(U, X_+ \wedge T^n)_\Phi)$ ,  $\cup_{i \in I} Seg(\Gamma_m(U, X_+ \wedge T^n)_\Phi)$  of the  $S^1$ -spectrum  $Seg(\Gamma_{\mathcal{A}}^f)$ . One has an equality of the  $S^1$ -subspectra*

$$\cup_{a \in (\mathcal{A} - *)} Seg(\Gamma_m(U, X_+ \wedge T^n)_a) = \cup_{\Phi \in I} Seg(\Gamma'_m(U, X_+ \wedge T^n)_\Phi)$$

and the inclusion

$$\begin{aligned} \cup_{a \in (\mathcal{A} - *)} Seg(\Gamma_m(U, X_+ \wedge T^n)_a) &= \cup_{\Phi \in I} Seg(\Gamma'_m(U, X_+ \wedge T^n)_\Phi) = Seg(\Gamma'_m(U, X_+ \wedge T^n)) \hookrightarrow \\ &\hookrightarrow \cup_{\Phi \in I} Seg(\Gamma_m(U, X_+ \wedge T^n)_{A(\Phi)}) = Seg(\Gamma_m(U, X_+ \wedge T^n)) \end{aligned}$$

is a stable equivalence of the  $S^1$ -spectra.

Set,

$$Fr_m^{S^1}(U, X_+ \wedge T^n) = Seg(\Gamma_m(U, X_+ \wedge T^n)) \text{ and } Fr_m^{S^1}(U, X_+ \wedge T^n)_\Phi = Seg(\Gamma_m(U, X_+ \wedge T^n)_\Phi),$$

$$F_m^{S^1}(U, X_+ \wedge T^n) = Seg(\Gamma'_m(U, X_+ \wedge T^n)) \text{ and } F_m^{S^1}(U, X_+ \wedge T^n)_\Phi = Seg(\Gamma'_m(U, X_+ \wedge T^n)_\Phi).$$

Under this notation the preceding proposition implies the following

**Theorem B.10.** *Let  $U, X \in \mathcal{S}m/k$  and let  $m, n \geq 0$  be integers. One has an equality of  $S^1$ -subspectra*

$$\bigvee_{\Psi \in (\mathcal{F}_m(U, X_+ \wedge T^n) - 0_m)} \mathcal{F}_m^{S^1}(U, X_+ \wedge T^n)_\Psi = \bigcup_{\Phi \in (\mathcal{F}_m(U, X_+ \wedge T^n) - 0_m)} \mathcal{F}_m^{S^1}(U, X_+ \wedge T^n)_\Phi$$

*of the spectra  $\mathcal{F}_m^{S^1}(U, X_+ \wedge T^n)$  and the inclusion*

$$\mathcal{F}_m^{S^1}(U, X_+ \wedge T^n) \subset \mathcal{F}_m^{S^1}(U, X_+ \wedge T^n) \quad (15)$$

*is a stable equivalence of the  $S^1$ -spectra.*

We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* This follows from Theorem B.10. Indeed, consider the composite morphism of  $S^1$ -spectra

$$\mathbb{Z}\mathcal{F}_m^{S^1}(U, X_+ \wedge T^n) \rightarrow \mathbb{Z}\mathcal{F}_m^{S^1}(U, X_+ \wedge T^n) \xrightarrow{\lambda_{X_+ \wedge T^n}} \mathbb{Z}\mathcal{F}_m^{S^1}(U, X_+ \wedge T^n),$$

where the left arrow is induced by the arrow (15). Within Definitions 2.4 and B.6, Theorem B.10 implies the left arrow is a stable equivalence of  $S^1$ -spectra. Note that the composite morphism is the identity map. Thus the morphism  $\lambda_{X_+ \wedge T^n}$  is a stable equivalence of  $S^1$ -spectra. This finishes the proof.  $\square$

## REFERENCES

- [AGP] A. Ananyevskiy, G. Garkusha, I. Panin, Cancellation theorem for framed motives of algebraic varieties, preprint arXiv:1601.06642.
- [BF] A. K. Bousfield, E. M. Friedlander, Homotopy theory of  $\Gamma$ -spaces, spectra, and bisimplicial sets, In Geometric applications of homotopy theory II, Lecture Notes in Math. 658, Springer Verlag, 1978, pp. 80-130.
- [GP1] G. Garkusha, I. Panin, Framed motives of algebraic varieties (after V. Voevodsky), preprint arXiv:1409.4372.
- [GP2] G. Garkusha, I. Panin, Homotopy invariant presheaves with framed transfers, preprint, 2015, arXiv:1504.00884.
- [GP3] G. Garkusha, I. Panin, Linear framed motives of algebraic varieties, preprint, 2016.
- [Mi] J. Milne. Étale cohomology, Princeton Mathematical Series 33, Princeton University Press, 1980.
- [Sch] S. Schwede, An untitled book project about symmetric spectra, available at [www.math.uni-bonn.de/people/schwede/SymSpec-v3.pdf](http://www.math.uni-bonn.de/people/schwede/SymSpec-v3.pdf) (version April 2012).
- [Voe] V. Voevodsky, Notes on framed correspondences, unpublished, 2001. Also available at [math.ias.edu/vladimir/files/framed.pdf](http://math.ias.edu/vladimir/files/framed.pdf)

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA SA2 8PP, UNITED KINGDOM

*E-mail address:* [g.garkusha@swansea.ac.uk](mailto:g.garkusha@swansea.ac.uk)

DEPARTMENT OF MATHEMATICS AND STATISTICS, 585 KING EDWARD AVENUE, OTTAWA ON K1N 6N5, CANADA

*E-mail address:* [alexander.neshitov@gmail.com](mailto:alexander.neshitov@gmail.com)

ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS, UNIVERSITETSKY PROSPEKT, 28, 198504, PETERHOF, ST. PETERSBURG, RUSSIA

*E-mail address:* [paniniv@gmail.com](mailto:paniniv@gmail.com)